Wheeling: A Diagrammatic Analogue of the Duflo Isomorphism

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Abstract

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We construct and prove a diagrammatic version of the Duflo isomorphism between the invariant subalgebra of the symmetric algebra of a Lie algebra and the center of the universal enveloping algebra. This version implies the original for metrized Lie algebras (Lie algebras with an invariant non-degenerate bilinear form). As an application of this isomorphism, we will compute the Kontsevich integral of the unknot and the Hopf link to all orders.

At the core of the proof, we use an elementary property of the Hopf link which can be summarized by the equation "$1 + 1 = 2$" in abacus arithmetic: doubling one component of the Hopf link is equivalent to taking the connected sum of two Hopf links. This property of the Hopf link turns out, when suitably interpreted, to be exactly the property required for the Duflo map to be multiplicative.

To compute the Kontsevich integral of the unknot, we use a property of the unknot that can be summarized by "$n{\cdot}0 = 0$": the n-fold connected cabling of the unknot is again an unknot.

Vaughan Jones
Dissertation Committee Chair

Contents

Acknowledgements

The proof of the Wheeling theorem (the Duflo isomorphism) was joint work with Dror Bar-Natan. The proof of the Wheels theorem (the Kontsevich integral of the unknot) was joint work with Thang T. Q. Le. I would like to thank these two collaborators, with whom it was a real pleasure to work. Many thanks also to Michel Duflo, Stavros Garoufalidis, Greg Kuperberg, Lev Rozansky, Michèle Vergne, and Pierre Vogel for many useful discussions, and especially to my advisor, Vaughan Jones.

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Chapter 1

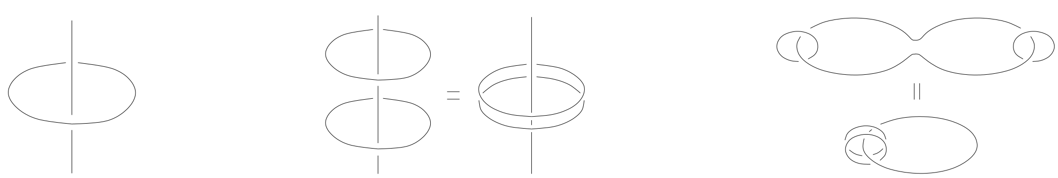
Introduction

1.1 Elementary knot theory

We begin by recalling two facts from elementary knot theory. These simple statements have deep consequences for Lie algebras and Vassiliev invariants. The two facts can be summarized by the catch phrases "$1 + 1 = 2$" and "$n \cdot 0 = 0$."

- "$1 + 1 = 2$." This refers to a fact in "abacus arithmetic." On an abacus, the number 1 is naturally represented by a single bead on a wire, as in Figure 1.1(a), which we think of as a tangle. The fact that $1 + 1 = 2$ then becomes the equality of the two tangles in Figure 1.1(b). On the left side of the figure, "$1 + 1$", the two beads are well separated, as for connect sum of links or multiplication of tangles; on the right side, "2", we instead start with a single bead and double it, so the two beads are very close together.

  In other terms, the connected sum of two Hopf links is the same as doubling one component of a single Hopf link, as in Figure 1.1(c).



(a) The link "1."

(b) "$1 + 1 = 2$."

(c) An alternate version of "$1 + 1 = 2$."

Figure 1.1: Elementary knot theory, part 1

- "$n \cdot 0 = 0$." In the spirit of abacus arithmetic, 0 is represented as just a single vertical strand. We prefer to close it off, yielding the knot in Figure 1.2(a). The knot $n \cdot 0$ is then this knot repeated n times, as in Figure 1.2(b).The two knots are clearly the same, up to framing.

1.2 The Duflo isomorphism

The first of these equations, "$1 + 1 = 2$", is related with the *Duflo isomorphism*. The Duflo isomorphism is an algebra isomorphism between the invariant part of the symmetric algebra and the center of the universal

(a) The knot "0."

(b) The knot "$n \cdot 0$," shown here for $n = 3$.

Figure 1.2: Elementary knot theory, part 2

enveloping algebra for any Lie algebra $\mathfrak{g}$. (The Poincaré-Birkhoff-Witt theorem gives a vector space isomorphism.) This isomorphism was first described for semi-simple Lie algebras by Harish-Chandra, and for all Lie algebras by Duflo [8].

Let us review briefly the Duflo isomorphism. Every Lie algebra $\mathfrak{g}$ has two associated algebras: the universal enveloping algebra $U(\mathfrak{g})$, generated by $\mathfrak{g}$ with relations $xy - yx = [x, y]$, and the symmetric algebra $S(\mathfrak{g})$, generated by $\mathfrak{g}$ with relations $xy - yx = 0$. There is a natural map between the two,

$$\chi : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}),$$

given by taking a monomial $x_1 \dots x_n$ in $S(\mathfrak{g})$ and averaging over the product (in $U(\mathfrak{g})$) of the x_i in all possible orders. By the Poincaré-Birkhoff-Witt (PBW) theorem, χ is an isomorphism of vector spaces and $\mathfrak{g}$-modules. Since $S(\mathfrak{g})$ is abelian and $U(\mathfrak{g})$ is not, χ is clearly not an algebra isomorphism. Even restricting to the invariant subspaces on both sides,

$$\chi : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{g})^{\mathfrak{g}} \cong Z(\mathfrak{g}),$$

χ is still not an isomorphism of algebras. The following theorem gives a modification that is an algebra isomorphism.

Theorem 1 (Duflo [8]). *For any finite-dimensional Lie algebra $\mathfrak{g}$, the map*

$$\Upsilon : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow Z(U(\mathfrak{g}))$$

is an algebra isomorphism, where

$$\Upsilon = \chi \circ \partial_{j^{\frac{1}{2}}}$$
$$j^{\frac{1}{2}}(x) = \det{}^{\frac{1}{2}} \left(\frac{\sinh(\frac{1}{2} \operatorname{ad} x)}{\frac{1}{2} \operatorname{ad} x} \right)$$

The notation of $\partial_{j^{\frac{1}{2}}}$ means to consider $j^{\frac{1}{2}}(x)$ as a power series on $\mathfrak{g}$ and plug in the (commuting) vector fields $\partial/\partial x^*$ on $\mathfrak{g}^*$. (Note that for $x^* \in \mathfrak{g}^*$, $\partial/\partial x^*$ transforms like an element of $\mathfrak{g}$). The result is an infinite-order differential operator on $\mathfrak{g}^*$, which we can then apply to a polynomial on $\mathfrak{g}^*$ ($\equiv$ an element of $S(\mathfrak{g})$). $j^{\frac{1}{2}}(x)$ is an important function in the theory of Lie algebras. Its square, $j(x)$, is the Jacobian of the exponential mapping from $\mathfrak{g}$ to the Lie group G.

1.3 Wheels

The bridges between the knot theory of Section 1.1 and the seemingly quite disparate Lie algebra theory of Section 1.2 are a certain spaces of uni-trivalent diagrams modulo local relations. ("Uni-trivalent" means that the vertices have valence 1 or 3; the 1-valent vertices are called the "legs" of the diagram.) On one

(a) A sample element of $\mathcal{A}$. (b) A sample element of $\mathcal{B}$.

Figure 1.3: Example web diagrams

hand, diagrams give elements of $U(\mathfrak{g})$ or $S(\mathfrak{g})$ for every metrized Lie algebra $\mathfrak{g}$ in a uniform way, as we will see in Chapter 2; on the other hand, they occur naturally in the study of finite type invariants of knots, as we will see in Chapter 3. Like the associative algebras associated to Lie algebras, these diagrams appear in two different varietes: $\mathcal{A}$, in which the legs have a linear order, as in Figure 1.3(a), and $\mathcal{B}$, in which the legs are unordered, as in Figure 1.3(b). As for Lie algebras, they each have a natural algebra structure (concatenation and disjoint union, respectively); and there is an isomorphism $\chi : \mathcal{B} \to \mathcal{A}$ between the two (averaging over all possible orders of the legs).

There is one element of the algebra $\mathcal{B}$ that will be particularly important for us: the "wheels" element of the title. It is the diagrammatic analogue of the function $j^{\frac{1}{2}}$ above.

$$\Omega = \exp \sum_{n=1}^{\infty} b_{2n} \omega_{2n} \in \mathcal{B}. \tag{1.1}$$

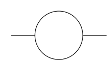
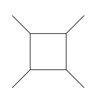
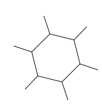
where

- The 'modified Bernoulli numbers' b_{2n} are defined by the power series expansion

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \log \frac{\sinh x/2}{x/2}. \tag{1.2}$$

  These numbers are related to the usual Bernoulli numbers $B_{2n} = 4n \cdot (2n)! \cdot b_{2n}$ and to the values of the Riemann ζ-function on the even integers. The first three modified Bernoulli numbers are $b_2 = 1/48$, $b_4 = -1/5760$, and $b_6 = 1/362880$.

- The '$2n$-wheel' ω_{2n} is the degree $2n$ web diagram made of a $2n$-gon with $2n$ legs:

$\omega_2 =$  , $\omega_4 =$  , $\omega_6 =$  , $\ldots$. (1.3)

Our main theorem is written in terms of ∂_Ω, the operation of applying Ω as a differential operator, which takes a diagram D and attaches some of its to Ω. (See Section 2.6 for the precise definition.)

Theorem 2 (Wheeling; joint with D. Bar-Natan). *The map $\Upsilon = \chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{A}$ is an algebra isomorphism.*

(All the notation above, including the definitions of $\mathcal{A}$ and $\mathcal{B}$, is explained in Chapter 2. The proof of the theorem is in Chapter 5.)

Although Theorem 2 was motivated by Lie algebra considerations when it was first conjectured [4, 7], the proof we will give, based on the equation "$1+1=2$" from Section 1.1, is entirely independent of Lie algebras and is natural from the point of view of knot theory. In particular, we obtain a new proof of Theorem 1 for metrized Lie algebras, with some advantages over the original proofs by Harish-Chandra, Duflo, and Cartan: our proof does not require any detailed analysis of Lie algebras, and so works in other contexts in which there is a Jacobi relation. For instance, our proof works for super Lie algebras with no modification.

Theorem 2 has already seen several applications. In Chapter 6 we will use it to compute the Kontsevich integral of the unknot, using our second elementary knot theory identity "$n \cdot 0 = 0$".

Theorem 3 (Wheels; joint with T. Le). *The Kontsevich integral of the unknot is*

$$Z(\bigcirc) = \Omega \in \mathcal{B}.$$

We also compute the Kontsevich integral of the Hopf link ⚭; which is intimately related to the map Υ above.

1.4 Related work

Theorem 2 was first conjectured by Deligne [7] and Bar-Natan, Garoufalidis, Rozansky, and Thurston [4], who also conjectured Theorem 3.

Further computations for a sizeable class of knots, links, and 3-manifolds (including all torus knots and Seifert-fiber homology spheres) have been done by Bar-Natan and Lawrence [5]. Hitchen and Sawon [12] have used Theorem 2 to prove an identity expressing the L^2 norm of the curvature tensor of a hyperkähler manifold in terms of Pontryagin classes. And in a future paper [19] I will show how to write simple formulas for the action of $\mathfrak{sl}_2(\mathbb{Z})$ on the vector space associated to a torus in the perturbative TQFT of Murakami and Ohtsuki [18].

These applications suggest that Theorem 2 is a fundamental fact. Another sign of fundamental facts is that they have many proofs. Besides the earlier work of Harish-Chandra, Duflo, and Cartan mentioned above, there are two other recent proofs of Theorem 2. One is due to Kontsevich [13, Section 8], as expanded by [2]. Kontsevich's proof is already at a diagrammatic level, similar to the one in this thesis, although it is more general: it works for all Lie algebras, not just metrized ones. His proof again uses a transcendental integral, similar in spirit to the one in Chapter 4. Another proof is due to Alekseev and Meinrenken [1]. The Alekseev and Meinrenken paper is not written in diagrammatic language, but seems to extend to the diagrammatic context without problems. Their proof does not involve transcendtal integrals: the only integral in their proof is in the proof of the Poincaré lemma (the homology of $\mathbb{R}^n$ is trivial in dimension > 0).

1.5 Plan of the thesis

The first few chapters are standard introductory material, included here to make this thesis as self-contained as possible. In Chapter 2, we review the diagrammatics of Lie algebras and define the space $\mathcal{A}$ and $\mathcal{B}$. In Chapter 3 we review the definition of Vassiliev invariants, again arriving at the same spaces $\mathcal{A}$ and $\mathcal{B}$; in addition, a few more spaces are defined in Section 3.3. The exposition is related to the standard exposition [3] and the work of Goussarov [9, 10] and Habiro [11], but is different from both. Chapter 4 is a review of the Kontsevich integral and its main properties, notably the behaviour under connect sum and cabling.

After this introductory material, we give the proof of Theorem 2 in Chapter 5, modulo fixing the coefficients b_{2n}, which is done in Appendix B. In Chapter 6, we prove Theorem 3, as well as similar results for the Hopf link.

Chapter 2

Diagrammatics of Lie algebras

Throughout this thesis, we will work heavily with a graphical notation for tensors, in particular inside a Lie algebra; in fact, this becomes more than just a notation for us. For the benefit of readers who may be unfamiliar with this notation, here is a quick refresher course.

2.1 Elementary tensors

In general, a tensor with n indices will be represented by a graph with n legs (or free ends). The indices of a tensor can belong to various vector spaces or their duals. Correspondingly, the legs of the graph should be colored to indicate the vector space and directed to distinguish between a vector space and its dual. See Figure 2.1 for some examples. Figure 2.1(a) shows a generic matrix $M \in \mathrm{Hom}(V, V) \simeq V^* \otimes V$. By convention, data flows in the direction of the arrows, so the incoming edge is the V^* factor and the outgoing arrow is the V factor. Figure 2.1(b) represents the bracket in a Lie algebra $[\cdot, \cdot] \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Because these vertices are so ubiquituous, these vertices will not be decorated. A representation R on a vector space V of a Lie algebra is a linear map $R \in \mathrm{Hom}(\mathfrak{g} \otimes V, V) \equiv \mathfrak{g}^* \otimes V^* n \otimes V$. This is depicted in Figure 2.1(c).

(a) A matrix M

(b) The bracket $[x, y]$ in a Lie algebra

(c) A representation of $\mathfrak{g}$ on V

(d) The product of two matrices M and N

Figure 2.1: Some elementary tensors

2.2 Composing Tensors

The use of tensors is that you can compose them in many different ways. For instance, two matrices $M, N \in Hom(V,V)$ can be multiplied:

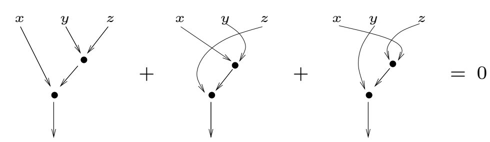
$$M \circ N = \sum_\beta M^\alpha_\beta N^\beta_\gamma$$

which we can represent graphically as in Figure 2.1(d). More generally, if a tensor has two indices with values in vector spaces that are dual to each other, they can be contracted; graphically, an incoming and outgoing leg of the same color can be connected.

As an important example, the Jacobi relation in a Lie algebra,

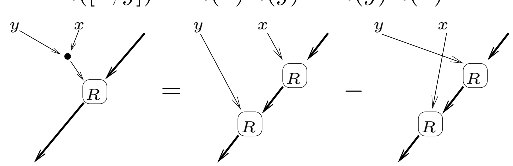
$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \qquad x,y,z \in \mathfrak{g}$$

can be expressed graphically as

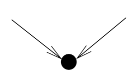
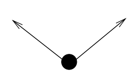


In the same way, the equation that a representation of a Lie algebra be a representation can be written graphically.

$$R([x,y]) = R(x)R(y) - R(y)R(x)$$



As another important case, suppose we have a metrized Lie algebra: that is, we have an invariant nondegenerate bilinear form on $\mathfrak{g}$. This gives us a tensor $(\cdot\,,\cdot) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ and its inverse in $\mathfrak{g} \otimes \mathfrak{g}$. Graphically, this gives us tensors


and


With the aid of these two tensors, we can glue two edges on which the arrows don't match. Furthermore, because they are inverses of each other, it doesn't matter if we stick in extra pairings. We will take this as license to drop all arrows on Lie algebra legs when working with metrized Lie algebras.

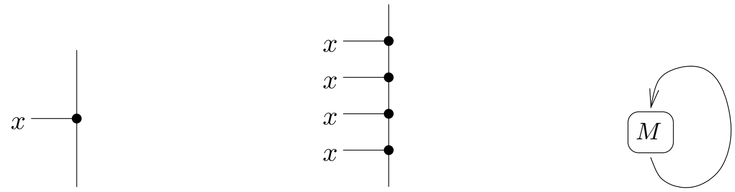
There is one point to be careful about: when we drop the decorations on a Lie bracket, we are assuming some symmetry of the bracket, since it is impossible to tell the bracket from a rotated version of itself. Fortunately, the Lie bracket is cyclicaly invariant when you identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using $(\cdot\,,\cdot)$ by invariance of the metric.

2.3 The function $j^{\frac{1}{2}}$

To gain practice with the graphical notation for tensors, let us find the graphical version of the function $j^{\frac{1}{2}}$ on $\mathfrak{g}$. Let us recall the definition of $j^{\frac{1}{2}}$:

$$j^{\frac{1}{2}}(x) = \det{}^{\frac{1}{2}} \frac{\sinh \frac{\operatorname{ad} x}{2}}{\frac{\operatorname{ad} x}{2}}.$$

We will work through this definition step by step.



(a) $\operatorname{ad} x$ as a matrix. (b) $(\operatorname{ad} x)^n$, for $n = 4$. (c) $\operatorname{tr} M$.

Figure 2.2: Building blocks of wheels

- $(\operatorname{ad} x)^n$. Here we take $\operatorname{ad} x$, considered as a matrix acting on $\mathfrak{g}$, and raise it to some power. Since $(\operatorname{ad} x)(y) = [x, y]$, the graphical representation of $\operatorname{ad} x$ is as in Figure 2.2(a). Note that the end labelled x is not considered as a leg in the tensor sense; instead, we plug in the (fixed) element $x \in \mathfrak{g}$ and get a fixed matrix. But see Section 2.4 for another point of view.

  To raise $\operatorname{ad} x$ to a power, just take several copies and string them together as in Figure 2.2(b).

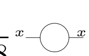
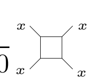
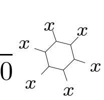
- Determinants and traces. The determinant of a matrix is a non-linear function of the matrix that does not fit well in our graphical notation. Fortunately, we can get rid of determinants using the equality

$$\det \exp M = \exp \operatorname{tr} M.$$

  The trace of a matrix is easy to understand graphically: take a matrix $M^a_b \in V^* \otimes V$ and sum over the diagonal, $a = b$; in other words, contract V with V^*. Graphically, we just connect the input of M with its output as in Figure 2.2(c).

We can now write a complete formula for $j^{\frac{1}{2}}$ in terms of graphs:

$$\begin{aligned} j^{\frac{1}{2}}(x) &= \det{}^{\frac{1}{2}} \frac{\sinh \frac{\operatorname{ad} x}{2}}{\frac{\operatorname{ad} x}{2}} \\ &= \exp(\frac{1}{2} \operatorname{tr}(\log \frac{\sinh \frac{\operatorname{ad} x}{2}}{\frac{\operatorname{ad} x}{2}})) \\ &= \exp(\frac{1}{2} \operatorname{tr}(\sum_{n=0}^{\infty} b_{2n} (\operatorname{ad} x)^{2n})) \\ &= \exp(\sum_{n=0}^{\infty} b_{2n} \omega_{2n}(x)) \end{aligned} \tag{2.1}$$

$$= \exp\Bigg(\frac{1}{48}$$  $$- \frac{1}{5760}$$  $$+ \frac{1}{362880}$$  $$- \cdots \Bigg)$$

The b_{2n} were defined in Equation 1.2. The ω_{2n} are as in Equation 1.3, with x placed on the legs.

2.4 The symmetric and universal enveloping algebras

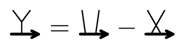
All the tensors we have drawn so far have had a fixed number of legs. We will also be interested in representing diagrammatically the symmetric algebra of a Lie algebra or its universal enveloping algebra so that we can, e.g., write $j^{1/2}(x)$ as an element of $S(\mathfrak{g})$ rather than just a function on $\mathfrak{g}$.

Recall that

$$S(\mathfrak{g}) = \bigoplus_{n \geq 0} \operatorname{Sym}^n(\mathfrak{g})$$

where $\operatorname{Sym}^n(\mathfrak{g})$ is the n'th symmetric power of $\mathfrak{g}$, which we take to be the quotient of $\mathfrak{g}^{\otimes n}$ by the symmetric group S_n. Diagrammatically, an element of $\operatorname{Sym}^n(\mathfrak{g})$ is easy to represent; just take a diagram with n legs (representing an element of $\mathfrak{g}^{\otimes n}$) and take the quotient by the symmetric group, i.e., forget about the labels on the legs. An element of $S(\mathfrak{g})$ is a diagram with any number of unlabelled legs.

For the universal enveloping algebra $U(\mathfrak{g})$, recall the definition: $U(\mathfrak{g})$ is the associative algebra generated by $\mathfrak{g}$, modulo the relation $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$. In particular, it is a quotient space of the free associative algebra on $\mathfrak{g}$, which is $\bigoplus_n \mathfrak{g}^{\otimes n}$. These are diagrams with n legs, with certain relations. We will remember these relations by placing the legs on a thick line, as in Figure 1.3(a). (Explicitly, we evaluate the interior part of the diagram to get an element in $\mathfrak{g}^{\otimes n}$; we then multiply the n elements of $\mathfrak{g}$ to get an element of $U(\mathfrak{g})$. The quotient means that the relation

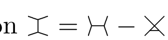
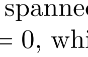


is satisfied.

2.5 Spaces of diagrams

Until now, the diagrams we have drawn have only been a particular notation for tensors in Lie algebras. In fact, the notation is more than a notation: they will be the fundamental objects of study. We will now define a space $\mathcal{B}$ that is informally the space of all tensors in $S(\mathfrak{g})$ that can be constructed from the Lie bracket, modulo the relations that can be deduced solely from the Jacobi identity. n

Definition 2.1. An *open Jacobi diagram* (variously called a Chinese Character, uni-trivalent graph, or web diagram) is vertex-oriented uni-trivalent graph, i.e., a graph with univalent and trivalent vertices together with a cyclic ordering of the edges incident to the trivalent vertices. Self-loops and multiple edges are allowed. The univalent vertices are called *legs*. In planar pictures, the orientation on the edges incident on a vertex is the clockwise orientation. Some examples are shown in Figure 1.3(b).

Definition 2.2. $\mathcal{B}^f$ is the vector space spanned by Jacobi diagrams modulo the IHX relation  and the anti-symmetry relation , which can be applied anywhere within a diagram.

Definition 2.3. The *degree* of a diagram in $\mathcal{B}^f$ is half the number of vertices (trivalent and univalent). $\mathcal{B}$ is the completion of $\mathcal{B}^f$ with respect to the grading by degree.

By the discussion above, any element of $\mathcal{B}^f$ gives an element in $S(\mathfrak{g})$ for any metrized Lie algebra $\mathfrak{g}$. In some sense, you can think of $\mathcal{B}$ as being related to a "universal (metrized) Lie algebra", incorporating information about all Lie algebras at once. But $\mathcal{B}$ is both bigger and smaller than that: the map to the product of $S(\mathfrak{g})$ for all metrized Lie algebras is neither injective nor surjective.

- There are elements of $\mathcal{B}$ that are non-zero but become zero when evaluated in any metrized Lie algebra. See Vogel [20] for details.
- Not all elements of $S(\mathfrak{g})$ are in the image of the map from $\mathcal{B}$ to $S(\mathfrak{g})$, as shown by the following lemma:

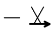
  Lemma 2.4. *Every element in the image of $\mathcal{B}$ is invariant under the action of any Lie group G (not necessarily connected) whose Lie algebra is $\mathfrak{g}$.*

  Proof. Every element in the image of $\mathcal{B}$ is made by contracting copies of the structure constants of the Lie algebra, which are invariant under G. ∎

There is a natural algebra structure on $\mathcal{B}$, the disjoint union $\cup$ of diagrams, which corresponds to the algebra structure on $S(\mathfrak{g})$.

We have similarly a diagrammatic analogue of $U(\mathfrak{g})$:

Definition 2.5. A *based Jacobi diagram* (also called chord diagram or Chinese Character diagram) is a Jacobi diagram with a total ordering on its legs. They are conventionally represented as in 1.3(a).

Definition 2.6. $\mathcal{A}^f$ is the vector space of based Jacobi diagrams modulo the Jacobi and antisymmetry relations as in $\mathcal{B}^f$, plus the STU relation  $=$ $-$.

Definition 2.7. The *degree* of a diagram in $\mathcal{A}^f$ is half the number of vertices. $\mathcal{A}$ is the completion of $\mathcal{A}^f$ with respect to the grading by degree.

There is likewise a natural algebra structure on $\mathcal{A}$: take two based Jacobi diagrams D_1, D_2 and place the legs of D_1 before the legs of D_2 in the total ordering on legs. For reasons that will become clear later, we call this the *connected sum* and denote it $D_1 \# D_2$.

These two products live on isomorphic spaces so may be confused. We usually write out the product in cases of ambiguity. If an explicit symbol for the product is omitted, the product is the disjoint union product $\cup$ unless otherwise specified.

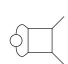
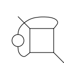
2.6 Diagrammatic Differential Operators

Our main goal in this thesis is to find a diagrammatic analogue of the Duflo isomorphism of Theorem 1. In that formula the differential operator $\partial_{j^{1/2}}$ plays a prominent role. We saw above how to write $j^{\frac{1}{2}}$ as an element of $\mathcal{B}$; but what does it mean to apply it as a differential operator?

Definition 2.8. For a diagram $C \in \mathcal{B}$ without struts (components like), the operation of *applying C as a differential operator*, denoted $\partial_C : \mathcal{B} \to \mathcal{B}$, is defined to be

$$\partial_C(D) = \begin{cases} 0 & \text{if } C \text{ has more legs than } D, \\ \text{the sum of all ways of gluing all the legs of } C \text{ to some (or all) legs of } D & \text{otherwise.} \end{cases}$$

For example,

$$\partial_{\omega_4}(\omega_2) = 0; \qquad \partial_{\omega_2}(\omega_4) = 8$$  $+\,4$  .

If C has k legs and total degree m, then ∂_C is an operator of degree $m - k$. By linear extension, we find that every $C \in \mathcal{B}$ without struts defines an operator $\partial_C : \mathcal{B} \to \mathcal{B}$. (We restrict to diagrams without struts to avoid circles arising from the pairing of two struts and to guarantee convergence: gluing with a strut lowers the degree of a diagram, and so the pairing would not extend from $\mathcal{B}^f$ to $\mathcal{B}$.)

We leave it to the reader to verify that this operation ∂_C is the correct diagrammatic analogue of applying C as a differential operator. (More precisely, these are constant coefficient differential operators; for instance, they all commute.) There are some good signs that we have the right definition:

- A diagram C with k legs reduces the number of legs by k, corresponding to a differential operator of order k.
- If $k = 1$ (C has only one leg), we have a Leibniz rule like that for linear differential operators:

  $$\partial_C(D_1 \cup D_2) = \partial_C(D_1) \cup D_2 + D_1 \cup \partial_C(D_2).$$

  (Actually, all diagrams with only one leg in $\mathcal{B}$ are 0, so we have to extend our space of diagrams slightly for this equation to be non-empty. Adding some extra vertices of valence 1 satisfying no relations is sufficient.)

- Multiplication on the differential operator side is the same thing as composition:

$$\partial_{C_1 \cup C_2} = \partial_{C_1} \circ \partial_{C_2}.$$

With a little more work, these properties can give a proof that this is the correct diagrammatic analogue of differential operators: extend the space of diagrams as suggested in the second point and add an operation of gluing two of these univalent ends. We will not give the details here.

2.7 The map Υ

We are now in a position to find the diagrammatic analogue of the Duflo map Υ we introduced for Lie algebras. Recall that Υ is the composition of two maps, the infinite-order differential operator $\partial_{j^{1/2}}$ and he Poincaré-Birkhoff-Witt isomorphism $\chi : S(\mathfrak{g}) \to U(g)$. Both of these have natural diagrammatic analogues.

From the computations in Section 2.3, the diagrammatic analogue of $j^{\frac{1}{2}}$ is the "wheels" element

$$\Omega = \exp(\sum_{m=0}^{\infty} b_{2m}\omega_{2m}) \in \mathcal{B}.$$

∂_Ω is the action of applying this element as a differential operator, as in the previous section; this is called the "wheeling" map.

The Poincaré-Birkhoff-Witt isomorphism is the isomorphism from $S(\mathfrak{g})$ to $U(\mathfrak{g})$ given by taking a monomial $x_1 \cdots x_n$ in $S(\mathfrak{g})$ to the average over the product in $U(\mathfrak{g})$ of the x_i in all possible orders. We can define a similar map diagrammatically.

Definition 2.9. The map $\chi : \mathcal{B} \to \mathcal{A}$ on a Jacobi diagram $D \in \mathcal{B}$ with n (unordered) legs is the average (not sum) in $\mathcal{A}$ over all possible orders on the legs.

As for Lie algebras, this is a vector space isomorphism. There are several different proofs of this fact. See Bar-Natan [3, Section 5.2, Theorem 8] for one.

The natural analogue of Theorem 1 (the Duflo isomorphism) is that the composition of these two maps be an algebra isomorphism; this is exactly Theorem 2:

Theorem 2 (Wheeling; joint with D. Bar-Natan). *The map $\Upsilon = \chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{A}$ is an algebra isomorphism.*

Note that ∂_Ω decreases or keeps fixed the number of legs of a diagram, and since Ω starts with 1, $\hat{\Omega}$ is lower-triangular with respect to the number of legs and so a vector space isomorphism. Since χ is also a vector space isomorphism, Υ is automatically bijective. The content of the theorem is that Υ is an algebra map:

$$\Upsilon(D_1 \cup D_2) = \Upsilon(D_1) \# \Upsilon(D_2). \tag{2.2}$$

Chapter 3

Vassiliev Invariants

The same spaces of diagrams that we saw appearing from Lie algebras in the last chapter (the spaces $\mathcal{A}$ and $\mathcal{B}$) also arise from 3-dimensional topology, via *Vassiliev* or *finite type* invariants. In this chapter we review this theory. Our presentation is slightly non-standard. It is related to the standard presentation of Bar-Natan [3] as well as the claspers or clovers of Goussarov [9,10] and Habiro [11]. But the differences are purely expositional; a reader familiar with the theory should feel free to skip this chapter after reviewing the definitions in Section 3.3.

All knots, tangles, etc. in this thesis are framed. The space $\mathcal{K}$ is the vector space (over $\mathbb{C}$ for convenience, though any field of characteristic 0 would work for this chapter) spanned by framed knots, i.e., smooth embeddings $S^1 \hookrightarrow \mathbb{R}^3$ with a normal vector field at each point of S^1, considered up to isotopy. More generally, suppose X is a compact 1-manifold, possibly with boundary. Then $\mathcal{K}(X)$ is the vector space spanned by framed tangles with skeleton X, i.e., framed smooth proper embeddings $X \hookrightarrow B^3$ with some fixed (but usually unspecified) behaviour at the boundary, considered up to isotopy relative to the boundary. In drawings, framings will be the blackboard framing: the framing perpendicular to the plane of the paper.

3.1 Definition of Vassiliev invariants

The idea of finite type invariants is to find invariants of tangles analagous to polynomial functions on vector space. A polynomial p of degree n on a vector space V can be defined to be a function that vanishes when you take the alternating sum of its values on the vertices of a parallelepiped:

$$p \text{ of degree } n \Leftrightarrow \sum_{T\subset S}(-1)^{|T|}p(x_0+\sum_{v\in T}v)=0, \qquad x_0\in V, S\subset V, |S|=n+1.$$

Similarly, a finite type invariant of degree n is a knot invariant that vanishes when you take the alternating sum on the vertices of a "cube of knots", determined by picking a knot projection and flipping some subset of the crossing.

Definition 3.1 (First version). A knot invariant $f : \mathcal{K} \to \mathbb{C}$[1] is *finite type* of degree n if, for every knot projection K with and subset S of the crossings and arcs of K, $|S| = n + 1$,

$$\sum_{T\subset S}(-1)^{|T|}f(K_T)=0,$$

where K_T is the knot obtained from K by flipping the crossings in T and adding a positive full twist to the framing along the arcs in T.

[1] More generally, we can allow values in an arbitrary abelian group

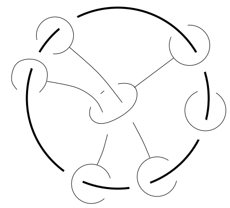


Figure 3.1: A sample 4-circus.

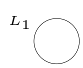
We need to allow the framing changes as a special case because crossing changes can only change the framing by two full twists. This definition, like all others in this section, extends without change to X-tangles for an arbitrary 1-manifold X.

Vassiliev invariants of degree 0 are constansts. In degree 1, we have the *linking number*, which is an invariant of 2-component links $X = \bigcirc_a \bigcirc_b$. Intuitively, this measures the number of times the component a winds around the component b. There are several ways to define the linking number precisely. One is to count, with signs, the number of times the component a crosses over the component b in a given projection. More intrinsically, there is a map from $S^1 \times S^1$ to S^2 which takes a point in $S^1 \times S^1$ to the normalized vector pointing from the point on a to the point on b. The linking number is the degree of this map. When you change a crossing in a projection, the linking number changes by 0 or ± 1, depending only on the intrinsic topology of the crossing (whether the crossing was a crossing of a component with itself or between two different components); when you change a second crossing, this difference doesn't change, so the linking number is finite type of degree 1.

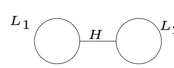
As a related example, every knot has a canonical framing given by the boundary of a Seifert surface; the difference between the given framing of a knot and its canonical framing is again finite type of degree 1.

Definition 3.1 is slightly unsatisfactory since it depends on picking a knot projection, which squashes 3 dimensions down to 2 in a slightly uncomfortable fit. We can fix this by "delocalising" the crossing change.

Definition 3.2. An *n-circus* for a knot K is an embedding in $S^3 \setminus K$ of n disjoint *single lassos* resp. *double lassos*



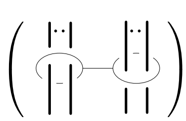
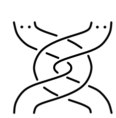
resp.



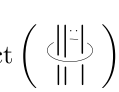
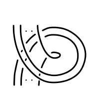
,

that are embedded trivially in S^3 in the absence of K.

Definition 3.3. The *action* Act_C on an n-circus C is the move

Act  =  (3.1)

on each of its component double lassos and the move

Act  =  (3.2)

on each of its component single lassos.

The geometric picture for the double lasso action is that the double lassos is like a kayak paddle. For those unfamiliar with kayaking, a kayak paddle has a blade[2] on either end. The blades at the two ends are at a 90° angle to each other. For us, the loops run around the outside of the blades the strands pass perpendicularly

[2]The part of a paddle that goes in the water.

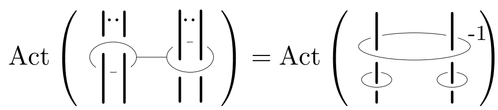
through the loops. Then you can shrink the handle down to zero and pass the strands through each other with no ambiguity, since they meet at a right angle. Unfortunately, this is hard to draw in a 2-dimensional picture. For the drawings in Equation 3.1 to really correspond to this geometric picture, we would need to include an extra (negative) quarter-twist in the handle of the lasso, but this would clutter future diagrams too much. The conventions here agree with those of Habiro [11], but are opposite those of Goussarov [9,10].

Definition 3.4 (Second version). A knot invariant f is *finite type* of degree n if, for every $n+1$-circus S,

$$\sum_{T\subset S}(-1)^{|T|}f(\mathrm{Act}_T(K))=0.$$

The formal linear combination of knots $\sum_{T\subset S}(-1)^{|T|}\,\mathrm{Act}_T(K)$ is called the *resolution* $\delta^{(n+1)}(S)$ of S.

Remark 3.5. A double lasso can be made out of three single lassos:



(where the "-1" indicates a single lasso to be applied in the reverse sense), so we could have have omitted double lassos from Definition 3.2 without changing the definition of finite type. But we prefer not to do this: instead, we will write single lassos in terms of double lassos. (See Relation 3.) The relationship between single and double lassos is comparable to the relationship between quadratic and bilinear forms.

3.2 Weight systems

The resolution of an $n+2$-circus is the difference of the resolutions of two $n+1$-circuses, so a Vassiliev invariant of degree n is also a Vassiliev invariant of degree m for $m>n$; i.e., Vassiliev invariants form an increasing filtration. The image of a Vassiliev invariant in the associated graded space is called its *weight system*; it is comparable to the highest degree term of a polynomial or the symbol of a differential operator. In this section we will try to identify this associated graded space. It is more convenient to work with the dual picture.

Definition 3.6. The n'th term of the *Vassiliev filtration* $\mathcal{K}=\mathcal{K}_0\supset\mathcal{K}_1\supset\cdots\supset\mathcal{K}_n\supset\cdots$ on the space of knots is the span of resolutions of n-component circuses in the complement of a knot.

Let $\mathrm{Diag}(C)$ or the *diagram* associated to C be the image of $\delta^{(n)}(C)$ in $\mathcal{K}_n/\mathcal{K}_{n+1}$.

Relation 1 (Homotopy). $\mathrm{Diag}(C)\equiv\mathrm{Diag}(C')$, where C and C' differ by homotopy of the knot and handles of the lasso (fixing the endpoints) in the complement of the loops of lassos.

Proof. Given an n-circus C, consider the $n+1$-circus $\tilde{C}$ with an additional double lasso which encircles two knot strands and/or handles of double lassos. The resolution of $\tilde{C}$ along the new double lasso is equal to the difference between C and a variant C' in which the two strands have crosses each other. Since $\tilde{C}$ has $n+1$ lassos, this difference is zero in the associated graded space $\mathcal{K}_n/\mathcal{K}_{n+1}$ so $\mathrm{Diag}(C)=\mathrm{Diag}(C')$. Likewise framing changes on a strand can be achieved by adding an additional single lasso looping it. These two moves generate homotopy as in the statement. ☐

With the Homotopy Relation, we have forgotten much of the topology of the embedding of an n-circus. The next relation lets us simplify the loops of double lassos.

Relation 2 (Splitting). Let $C,C_1,\ldots,C_k$ be n-circuses which are the same except for one loop of one double lasso. For that loop, suppose that the loop of C is homotopically the sum of the loops of $C_1,\ldots,C_k$. Then $\mathrm{Diag}(C)=\mathrm{Diag}(C_1)+\cdots+\mathrm{Diag}(C_k)$.

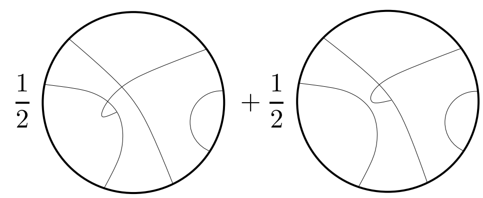
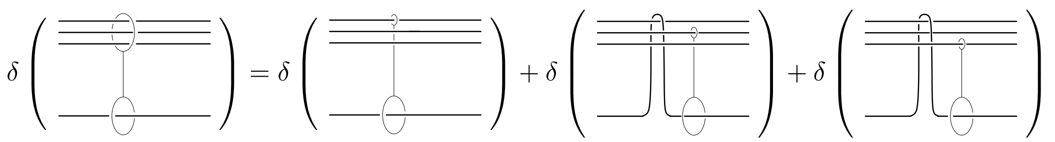


Figure 3.2: The diagrams corresponding to Figure 3.1.

Proof. A multiple crossing can be done step by step:

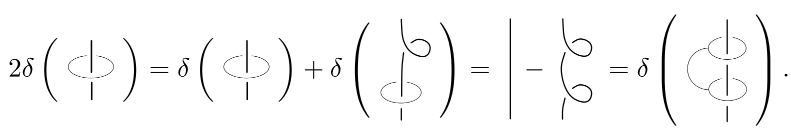


(δ is the operation of resolving the double lasso.) Modulo Relation 1, we can forget the extra little hooks. □

Relation 3 (Single lasso). Let C be a circus with a single lasso and C' be the same circus with a the single lasso replaced by a double lasso whose two loops are parallel to the old single lasso. Then

$$2\,\mathrm{Diag}(C) = \mathrm{Diag}(C').$$

Proof.



□

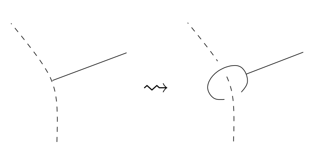
Let us see what the space of diagrams $\mathcal{K}_n/\mathcal{K}_{n+1}$ is, using only Relations 1, 2, and 3. By Relation 3, we can get turn all single lassos into double lassos. By Relation 2, we can make sure each loop of a double lasso encloses only a single strand (either a strand of the knot or a handle of a double lasso); and by Relation 1 only the combinatorics of these loops and connections are relevant. So we can reduce any diagram $\mathrm{Diag}(C)$ to a sum of encircled, enriched Jacobi diagrams as in Figure 3.2.

Definition 3.7. An *encircled Jacobi diagram* is a vertex-oriented trivalent graph with a distinguished cycle, called the *skeleton*. As a special case, a distinguished circle with no vertices is allowed. Vertices on the distinguished circle are called *external*; other vertices are called *internal*.

Definition 3.8. An *routed Jacobi diagram* is a Jacobi diagram with two of the vertices incident to each internal vertex distinguished.

In drawings, the distinguished cycle is indicated by thick lines. Thin lines are double lassos and dashed lines are either of the two. For routed Jacobi diagrams, the two distinguished edges are continuous and the remaing edge meets them at right angles.

Most routed encircled Jacobi diagrams correspond to n-circuses in a knot complement by embedding the diagram arbitrarily in $\mathbb{R}^3$, resolving each vertex like this



The distinguished cycle becomes the knot. In good cases, this yields an n-circus in the knot complement. We will examine this situation in Appendix A, where we will prove the following proposition.

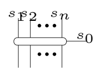
Definition 3.9. A Jacobi diagram is *boundary connected* if there are no connected components disjoint from the skeleton (i.e., the distinguished cycle in the case of encircled Jacobi diagrams).

Proposition 3.10. *Any boundary connected Jacobi diagram has a routing which corresponds to an n-circus. Furthermore, the diagrams of all such n-circuses are equal.*

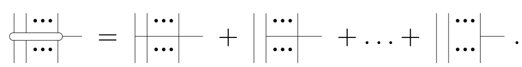
Proposition 3.10 gives us license to forget the routing on Jacobi diagrams so long as we restrict to boundary connected diagrams.

The relations above have more consequences. Any diagram containing an empty loop is 0 by the Splitting relation. But this empty loop need not be obviously empty in the associated Jacobi diagrams. As the loop is deformed, it can pass over strands or vertices; these give us two types of relations. Before we introduce the new relations, let us define a handy graphical notation.

Definition 3.11. A rounded box like

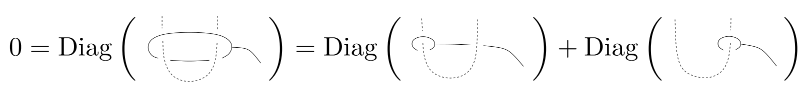


is, by definition, the sum over the n ways of attaching the strand s_0 to one of the strands $s_1, \ldots, s_n$:

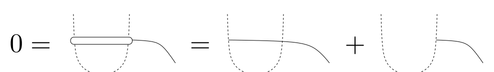


The relations are now

1. (Antisymmetry) We can push a strand (either a piece of knot or a handle) through the loop of an empty lasso:

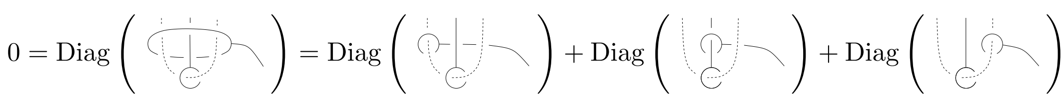

(3.3)

or

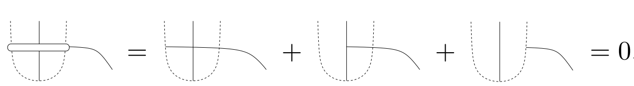


In other words, reversing the orientation of a trivalent vertex negates the diagram.

2. (vertex) We can push a vertex through the loop of a lasso:


(3.4)

or



As with the antisymmetry relation, this comes in two different versions, depending on whether the thin line is the handle of a lasso or a strand of the knot. The relation is also called the Jacobi or IHX relation in the first case, and the universal enveloping algebra or STU relation in the second case.

Our approximation to the space of diagrams $\mathcal{K}_n/\mathcal{K}_{n+1}$ is the space $\mathcal{A}_n(\bigcirc)$ of degree n boundary-connected encircled Jacobi diagrams modulo the antisymmetry and vertex relations. We have shown that the map $\mathrm{Diag} : \{n\text{-circuses}\} \to \mathcal{K}_n/\mathcal{K}_{n+1}$ factors through a map Diag' to $\mathcal{A}_n(\bigcirc)$:

$$\mathrm{Diag} : n\text{-circuses} \xrightarrow{\mathrm{Diag}'} \mathcal{A}(\bigcirc) \xrightarrow{m} \mathcal{K}_n/\mathcal{K}_{n+1}$$

So $\mathcal{A}_n(\bigcirc)$ is an upper bound for $\mathcal{K}_n/\mathcal{K}_{n+1}$. The map m is actually an isomorphism, but this fact seems to be difficult to prove directly. It follows from the construction in Chapter 4.

3.3 Spaces of diagrams, redux

As mentioned previously, all constructions in the previous two sections extend without change to the more general context of tangles with skeleton X, for X any 1-manifold. Repeating the discussion of Section 3.2, we are naturally led to following definition for the corresponding spaces of diagrams.

Definition 3.12. For a compact 1-manifold X (possibly with boundary), a *Jacobi diagram based on* X is a vertex-oriented uni-trivalent graph Γ together with an embedding $\phi : X \hookrightarrow \Gamma$ up to isotopy so that $\phi(\partial X)$ is exactly the univalent vertices of Γ. X is the *skeleton* of Γ. The *degree* of such a diagram is half the number of trivalent vertices of Γ.

Definition 3.13. For a compact 1-manifold X, the space $\mathcal{A}^f(X)$ is the vector space spanned by Jacobi diagrams based on X modulo the antisymmetry and vertex relations of Equations 3.3 and 3.4. $\mathcal{A}(X)$ is the completion of $\mathcal{A}^f(X)$ with respect to the degree.

Definition 3.14. $\mathcal{A}^{\mathrm{bc}}(X)$ is the subspace of $\mathcal{A}(X)$ spanned by boundary-connected Jacobi diagrams: diagrams with no connected components that do not contain a portion of the skeleton X.

By the construction of Chapter 4, $\mathcal{A}^{\mathrm{bc}}(X)$ is isomorphic to the space $\mathcal{K}_n(X)/\mathcal{K}_{n+1}(X)$.

To relate these definitions to the definitions in Section 2.5, note that $\mathcal{A}(\uparrow)$ (i.e., $\mathcal{A}$ of an oriented interval) is isomorphic to $\mathcal{A}$. Recall that $\mathcal{A}$ was defined to be the space of Jacobi diagrams with some ordered, univalent ends, modulo the antisymmetry, Jacobi, and STU relations. To pass from a diagram in $\mathcal{A}$ to a diagram in $\mathcal{A}(\uparrow)$, place the univalent vertices on the interval in the specified order, always attaching from the left side of the interval (as in Figure 1.3(a)). The antisymmetry relation in $\mathcal{A}$ becomes an antisymmetry relation in $\mathcal{A}(\uparrow)$ and the Jacobi and STU relations both become vertex relations (of the two different types) in $\mathcal{A}(\uparrow)$. The inverse map from $\mathcal{A}(\uparrow)$ to $\mathcal{A}$ is applying antisymmetry relations so all legs attach from the left and dropping the interval, remembering the order. In the future we will not distinguish between these two spaces.

We will also use versions of the space $\mathcal{B}$ in this more general context. Recall that $\mathcal{B}$ is the space $\mathcal{A}$ with the ordering on legs (and the STU relation) dropped. We can perform a similar operation on any interval component of X.

Definition 3.15. A *Jacobi diagram based on* $X \cup Y$, where X is a 1-manifold and Y is a set of asterisks $*$, is a vertex-oriented uni-trivalent graph Γ with a proper embedding $\iota : X \to \Gamma$ up to isotopy with the univalent vertices that are not in the image of ∂X labelled by asterisks in Y. In addition to self-loops and multiple edges, circle components with no vertices are allowed, as long as they are in the image of ι. The *degree* of Γ is half the number of trivalent plus Y-labelled univalent ends.

Definition 3.16. The space $\mathcal{A}^f(X \cup Y)$, X and Y as above, is the space of Jacobi diagrams on $X \cup Y$ modulo antisymmetry relations at each trivalent vertex and vertex relations around each trivalent vertex. $\mathcal{A}(X \cup Y)$ is the completion of $\mathcal{A}^f(X \cup Y)$ with respect to degree.

For X a collection of 1-manifolds and/or asterisks, there is a map $\chi_x : \mathcal{A}(*_x \cup X) \to \mathcal{A}(\uparrow_x \cup X)$, defined analogously to Definition 2.9: take the average over all possible ways of ordering the univalent legs labelled by x and attach them to an oriented interval on the left side. The proofs that χ is a vector space isomorphism also prove that χ_x is a vector space isomorphism.

If we want to similarly symmetrize over circle components, we run into a snag. There is a natural map from $\mathcal{A}(| \cup X)$ to $\mathcal{A}(\bigcirc \cup X)$ given by attaching the two univalent ends in the diagram that are the images of the endpoints of the interval $|$. In the simplest case this map is an isomorphism.

Lemma 3.17. *The spaces* $\mathcal{A}(\uparrow)$ *and* $\mathcal{A}(\circlearrowright)$ *are isomorphic.*

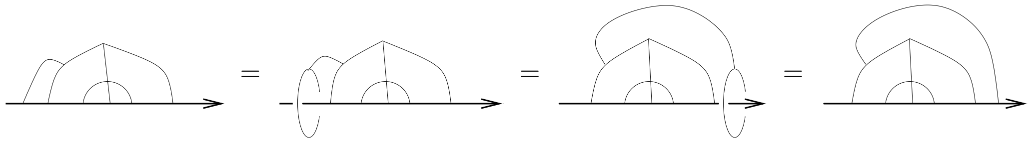


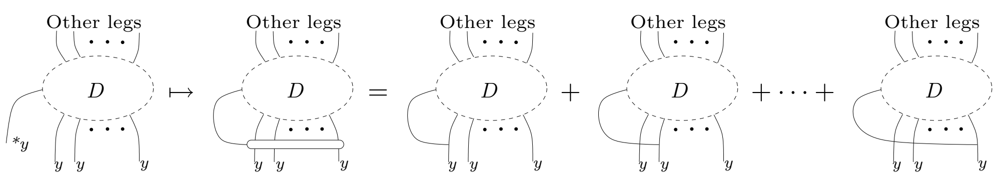
Figure 3.3: A leg in $\mathcal{A}(|)$ can be moved from one end of the interval to the other.

Proof. The map from $\mathcal{A}(\uparrow)$ to $\mathcal{A}(\circlearrowright)$ defined above is clearly surjective. To see that it is injective, we need to check that diagrams differing by cyclic permutations are already equal in $\mathcal{A}(\uparrow)$. It suffices to check that two Jacobi diagrams based on an interval differing by moving a single leg from the beginning to the end of the interval are equal modulo the antisymmetry and vertex relations. Since vacuum diagrams play no part in this problem, let us assume that there are none. We can then pick a good routing of the diagram and consider an associated n-circus. Now we can expand the first loop on the interval, pass it over the rest of the circus, and shrink it down again; we have moved this leg from the beginning to the end of the interval. See Figure 3.3.

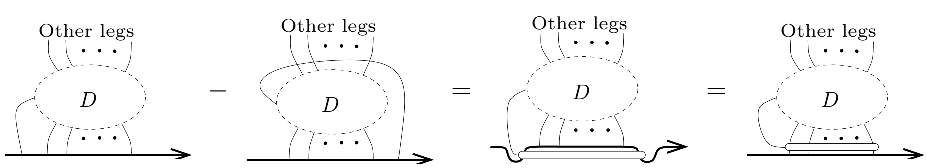
This can also be achieved by a sequence of vertex and antisymmetry relations without referring to topology by replacing the loops in Figure 3.3 by rounded boxes as in Definition 3.11 and sweeping the boxes from one end of the interval to the other, applying vertex and antisymmetry relations along the way. ☐

The same proof works to show that $\mathcal{A}(\uparrow \cup X) \simeq \mathcal{A}(\circlearrowright \cup X)$ where X is a closed 1-manifold; but the proof does not work (and the statement is not true) if there is another interval component. Explicitly, $\mathcal{A}(\uparrow\uparrow) \not\simeq \mathcal{A}(\uparrow\circlearrowright) \simeq \mathcal{A}(\circlearrowright\circlearrowright)$. The problem is that as we sweep the loops they get "caught up" on the extra ends of X. So if we want to symmetrize over the legs on circle components in general, we need to add a new relation.

Definition 3.18. In $\mathcal{A}(*_y \cup X)$, *link relations* on y are parametrized by Jacobi diagrams based on $*_y \cup X$ in which one of the y-labelled legs is distinguished and marked by '$*y$'. The corresponding link relation is the sum of all ways of attaching the marked leg to all the other legs labelled y:



Link relations are the image in $\mathcal{A}(*_y \cup X)$ of moving one strand from the beginning to end of the interval in $\mathcal{A}(\uparrow_y \cup X)$:



Symmetrizing the last diagram leaves it in the same form. So, inside $\mathcal{A}(\circlearrowright_y \cup X)$, all link relations vanish. Conversely, if we take the quotient of $\mathcal{A}(*_y \cup X)$ by all link relations on y, we get a space isomorphic to $\mathcal{A}(\circlearrowright_y \cup X)$.

Definition 3.19. $\mathcal{A}(X \cup Y \cup Y')$, where X is a 1-manifold, Y is a set of asterisks $*$, and Y' is a set of circled asterisks $\circledast$, is the space of Jacobi diagrams based on $X \cup Y \cup Y'$ modulo the vertex and antisymmetry relations as before and, in addition, link relations on each label in Y'.

Chapter 4

The Kontsevich integral and its properties

In this chapter, we will briefly review the *Kontsevich integral*, a remarkable invariant of tangles that is a universal Vassiliev invariant.

Definition 4.1. A *universal Vassiliev invariant* Z is a map $Z : \mathcal{K}(X) \to \mathcal{A}(X)$ so that, for any n-circus C in the complement of an X-tangle,

$$Z(\delta^{(n)}(C)) = \mathrm{Diag}'(C) + \text{higher order terms}$$

Recall that $\mathrm{Diag}' : \mathcal{K}(X) \to \mathcal{A}_n(X)$ is the approximation to $\mathrm{Diag} : \mathcal{K}(X) \to \mathcal{K}_n(X)/\mathcal{K}_{n+1}(X)$.

A universal invariant provides an inverse to the map $m : \mathcal{A}_n(X) \to \mathcal{K}_n(X)/\mathcal{K}_{n+1}(X)$ constructed in Section 3.2, and so proves that the two spaces are isomorphic. Furthermore, such an invariant is universal in the category-theoretical sense: any finite-type invariant with values in a vector space factors through Z.

We will not provide complete details here; for complete proofs, see [3, 16, 17]. Our main interest is in the good properties the Kontsevich integral satisfies under a few natural topological operations; we will consider connected sum and connected and disconnected cabling. Except for the computation in Section 4.5.3, which is due to Thang Le, all material in this chapter is standard.

4.1 The Kontsevich integral for braids

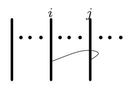
We will first define the Kontsevich integral for braids. A *braid* on n strands for our purposes is a smooth map from the interval $[0, 1]$ to the configuration space of n points in $\mathbb{R}^2$. There is a canonical framing for braids, since the vector field pointing in (say) the y direction is always normal to the braid. We can multiply two braids if the endpoint of one is the starting point of the other, just by placing one on top of the other and smoothing a little.

The Kontsevich integral for braids is the holonomy of a certain formal connection on the configuration space of n distinct points in the plane called the formal Knizhnik-Zamolodchikov (KZ) connection. It takes values in the space of chord diagrams appropriate to n-strand braids,

$$\mathcal{A}(\underbrace{|\cdots|}_{n\text{ copies}}) \stackrel{\text{def}}{=} \mathcal{A}(n).$$

Note that $\mathcal{A}(n)$ has an algebra structure given by vertical composition, just like braids. The values actually lie in the subspace of this space generated by horizontal chords.

Let c_{ij} be a chord connecting the i'th and j'th strands,

$c_{ij} =$ 

Let $z_j = x_j + iy_j$ be complex coordinates for the configuration space of n points. Then the connection 1-form of the KZ connection is

$$\mu_n = \frac{1}{2\pi i}\sum_{i<j} c_{ij} d\log(z_i - z_j) = \frac{1}{2\pi i}\sum_{i<j} c_{ij}\frac{dz_i - dz_j}{z_i - z_j}.$$

Exercise 4.2. Show that the Knizhnik-Zamolodchikov connection is flat:

$$d\mu_n + [\mu_n, \mu_n] = 0.$$

The Kontsevich integral of a braid is defined to be the holonomy of this connection, defined using Chen's iterated integral construction [6]

$$Z(B) = \sum_{m=0}^{\infty}\int_{0<t_1<\cdots<t_m<1} B^*(\mu_n)(t_1)\cdots B^*(\mu_n)(t_m).$$

This is invariant under isotopy of the braid since the Knizhnik-Zamolodchikov connection is flat.

More explicitly, we can write this as

$$Z(K) = \sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\int_{t_1<\ldots<t_m}\sum_{\substack{\text{applicable pairings}\\ P=\{(z_i,z_i')\}}} D_P \bigwedge_{i=1}^{m}\frac{dz_i - dz_i'}{z_i - z_i'} \in \mathcal{A}(n). \tag{4.1}$$

In the above equation,

- an 'applicable pairing' is a choice of an unordered pair (z_i, z_i') for every $1 \le i \le m$, for which (z_i, t_i) and (z_i', t_i) are distinct points on K.
- D_P is the chord diagram naturally associated with K and P, an appropriate product of the c_{ij}'s.
- Every pairing defines a map $\{t_i\} \mapsto \{(z_i, z_i')\}$ locally around the current values of the t_i's. Use this map to pull the dz_i's and dz_i''s to the space $t_1 < \cdots < t_m$ and then integrate the indicated wedge product over that simplex.

4.2 First properties: Universal and Grouplike

Let us first check that this invariant is, indeed, a universal finite-type invariant. The statement of universality has to be modified slightly for braids, since an action of an arbitrary circus will take us out of braids and into general n-strand tangles. We will restrict to *horizontal circuses*: circuses in which each double lasso lies in a horizontal (constant t) plane. Action on such a circus yields another braid. We have to show that Kontsevich integral of the resolution of a horizontal m-circus is the corresponding graph plus higher order terms.

Isotop the braid and the lassos so that all the lassos are very short, connecting strands close to each other. Also apply the splitting relation so that each loop of a lasso encloses a single strand. When we resolve a single lasso, we get a difference of two terms differing only near the crossing. The KZ connection will be constant on chords that do not touch the crossing and nearly constant on chords only one end of which reaches the crossing. For the alternating difference to be non-trivial, there must be at least one chord connecting the

two strands involved in the lasso. Similarly, if we resolve an entire circus, the integral will be constant or nearly constant if there is not at least one chord per lasso connecting the two strands of the lasso. Thus the lowest degree contribution to $Z(B_D)$ is at least degree m, and the degree m term is proportional to the desired chord diagram. To check the constant of proportionality, note that we are integrating $d(logz)/2\pi i$ on a counterclockwise circle around 0; by Cauchy's theorem, the answer is 1.

Next we will show that the Kontsevich integral is *grouplike*. For motivation, consider that the product of two finite type invariants v_1, v_2 is again finite type. The Kontsevich integral, as the universal finite type invariant, encodes the information about v_1, v_2, and $v_1 \cdot v_2$; the grouplike property encodes how the three invariants are related.

Definition 4.3. For any 1-manifold X, the coproduct $\square : \mathcal{A}(X) \to \mathcal{A}(X) \otimes \mathcal{A}(X)$ takes a Jacobi diagram D to the sum, over all partitions $D = D_1 \cup D_2$ of D into two parts in which both D_1 and D_2 contain X but are otherwise disjoint, of $D_1 \otimes D_2$. In other words, assign each connected component of $D \setminus X$ to the first or second tensor factor and sum over all possibilities.

Exercise 4.4. Let $v_1, v_2 : \mathcal{K} \to \mathbb{C}$ be Vassiliev invariants of degree n_1 with weight systems $w_i : \mathcal{A} \to \mathbb{C}$. ($w_i$ is supported on Jacobi diagrams of degree n_i.) Show that $v_1 \cdot v_2$ is a Vassiliev invariant of degree $n_1 + n_2$ and weight system

$$(w_1 \otimes w_2) \circ \square : \mathcal{A} \to \mathbb{C}$$

Hint 4.5. The evaulation of $v_1 v_2$ on the resolution of a k-circus can be written as a sum of terms

$$(v_1 \cdot v_2)(\delta^{(k)}(C)) = \sum v_1(\delta^{(l)}(C_1))v_2(\delta^{(k-l)}(C_2))$$

where C_1 and C_2 are appropriate circuses of degrees l and $k - l$. (Think about proving that the product of polynomials is polynomial.) For $k = n_1 + n_2 + 1$, at least one of the two factors vanishes. For $k = n_1 + n_2$, the sum divides the lassos of C into two subsets of size n_1 and n_2. For circuses related to graphs, each connected component must go into the same subset or the result is 0.

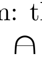
Proposition 4.6. *For any braid B, $Z(B)$ is* grouplike*:*

$$\square Z(B) = Z(B) \otimes Z(B).$$

Proof (Sketch). This is a general property of the holonomy of a connection with values in a Hopf algebra in which the connection form is primitive. (The usual example is the holonomy of a Lie-algebra valued connection, which takes values in the the Lie group or, alternatively, the grouplike elements inside the universal enveloping algebra.) $\square$

By the general structure theory of Hopf algebras, any time there is a multiplication on $\mathcal{A}(X)$ compatible with the comultiplication $\square$, a grouplike element is an exponential of primitive elements. Here a *primitive* element is an element x such that $\square(x) = x \otimes 1 + 1 \otimes x$. In particular, the multiplications in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}(n)$ are compatible with $\square$.

4.3 Cups and caps

In this thesis, we are not interested in braids, but rather in knots and, more generally, tangles. The difference is the presence of critical points, either minima $\cup$ or maxima $\cap$. The integral in Equation 4.1 naturally extends to this context. But there is a problem: the integral no longer converges. For instance, the integral corresponding to a single chord at a maximum , with the strands are a distance d apart at the bottom, is

$$\int_{z=d}^{0} \frac{1}{2\pi i} \frac{dz}{z} \cdot$$ 

which diverges logarithmically.

We can fix the divergence by terminating the integral when the points are a displacement ϵ apart and multiplying by the counterterm $\exp(-\frac{\log\epsilon}{2\pi i})\ominus$. It turns out that there is then a finite limit as $\epsilon \to 0$. To make the counterterm well-defined, we need to pick a branch of $\log\epsilon$ near each critical point of the height function. For concreteness, turn the knot near each critical point so that the tangent at the critical point is along the x axis (so ϵ is real) and take the real value of $\log|\epsilon|$ in the counter term. A knot with the critical points pinned down like this has a canonical framing, since a vector field pointing in the y direction is transverse to the knot. (If we project the knot onto the x-t plane, this is the blackboard framing.) If we twist a maximum counterclockwise or a minimum clockwise by π, the counter term changes by $\exp(\frac{1}{2}\ominus)$ and the above framing changes by 1 unit, so pinning down the critical points like this is equivalent to a choice of framing. See Le and Murakami [15] or Lescop [17] for details.

For any X-tangle T, let $\tilde{Z}(T)$ be the renormalized $\mathcal{A}(X)$ valued Kontsevich integral as above. Although $\tilde{Z}$ converges, it is not an invariant. In particular, $\tilde{Z}(\cap\!\!\cup) \neq \tilde{Z}(|) = |$. Let $\nu^{-1} = \tilde{Z}(\cap\!\!\cup)$.

Definition 4.7. The Kontsevich integral $Z(T)$ is $\tilde{Z}(T)$ multiplied by ν at each local maximum.

Now whenever we want to straighten a wiggle $\cap\!\!\cup$, there is an extra ν to cancel the ν^{-1} from the non-invariance of $\tilde{Z}(\cap\!\!\cup)$. With these definitions, $Z(\bigcirc) = \nu$. This element, ν, is what we will compute in Chapter 6.

It is straightforward to see that this extension of Z to tangles is still grouplike and a universal finite-type invariant.

From the discussion above about the framing and the critical points, we can compute what happens if we change the framing on a component.

Lemma 4.8. *Given a knot K, let K' be the same knot with the framing twisted by f positive twists. Then*

$$Z(K') = Z(K) \# \exp\left(\frac{f}{2}\ominus\right).$$

Similar results hold for changing the framing on a component in a tangle, where we multiply by $\exp(\frac{f}{2}\ominus)$ *on the affected component.* □

With only a little more work we can be more precise. The integrals contributing to a single chord $\ominus$ come from the term $m = 1$ in Equation 4.1. Since all such integrals are of the form $\frac{1}{2\pi i}\int\frac{dz}{z} = \frac{\log z}{2\pi i}$, this term can be computed. Mostly it computes the winding numbers of strands about themselves. Careful accounting of what happens at the maxima and minima shows that the coefficient of $\ominus$ is $1/2$ the number of positive crossings minus the number of negative crossings. If we combine this with the grouplike property of the Kontsevich integral, we get the following lemma.

Lemma 4.9. *The Kontsevich integral of a knot K with framing f relative to the canonical framing (the framing given by, e.g., a Seifert surface for K) is*

$$Z(K) = \exp_{\#}\left(\frac{f}{2}\frown\right) \# D \in \mathcal{B}$$

where D is strutless: has no connected components which are struts $\frown$. □

4.4 Connected sum

The first thing to notice about our definition of Z is that it is local in the horizontal plane: each horizontal slice is independent. Therefore, for any tangles T_1, T_2 so that the upper boundary of T_1 is the same as the lower boundary of T_2,

$$Z(T_1 \cdot T_2) = Z(T_1) \cdot Z(T_2) \tag{4.2}$$

Figure 4.1: A $(1,1)$ tangle and its closure.

where the multiplications of tangles on the LHS and web diagrams on the RHS are both vertical stacking: place the two objects on top of each other and join the corresponding legs.

We will be particularly interested in the case where the tangles are $(1,1)$ tangles, with one incoming and one outgoing strand. These are closely related to knots: we can turn a $(1,1)$ tangle canonically into a knot by joining the top to the bottom. See Figure 4.1 for an example. What does this do the the Kontsevich integral Z? By shrinking the tangle, the dotted box in Figure 4.1, the interactions between the dotted box and the original strand in the integral will be go to 0 while leaving the integral within the box unchanged. The differences between the invariants of the two are caused by (a) the correction introduced at the maximum and (b) changing the space of values by closing the strand, which by Lemma 3.17 is isomorphism. Letting T by the tangle and K be its closure, which could be a knot or a link with a distinguished component, we find

$$Z(K) = \nu \# Z(T) \tag{4.3}$$

where the two spaces are identified by their isomorphism.

Let T_1, T_2 be two $(1,1)$ tangles and K_1, K_2 their closures. Then

$$\begin{aligned} Z(K_1 \# K_2) &= Z(T_1 \cdot T_2) \# \nu \\ &= Z(T_1) \# Z(T_2) \# \nu \\ &= Z(K_1) \# Z(K_2) \# \nu^{-1}. \end{aligned}$$

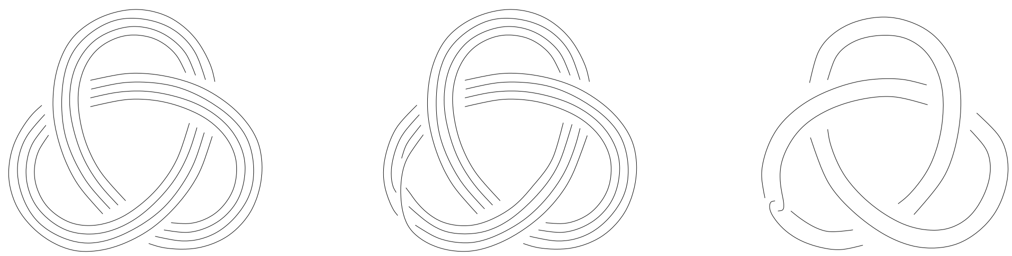
4.5 Cables

We now consider the operation of *cabling*: replacing a knot with n parallel copies of the same knot, as in Figure 4.2. There are two versions: the disconnected cabling, as in Figure 4.2(a), ending up with an n-component link, and the connected cabling, as in Figure 4.2(b), in which you add a twist so the result is again a knot.

Note that this operation is only well-defined for framed knots, and for connected cabling, you need to specify the extra twist to be added. It is natural to include this extra twist in the framing as a kind of "rational framing."

4.5.1 General Satellites

Let us start with some generalities about finite type invariants of satellites. A general satellite operation is specified by a link L embedded in a solid torus; the operation is to take a knot K, remove a tubular



(a) Disconnected cable.

(b) Connected cable.

(c) General satellite: the Whitehead double

Figure 4.2: Example cablings and satellites of the trefoil knot.

neighborhood of it, and glue in the solid torus with the embedded link, obtaining a new link K_L. An example of the result is in Figure 4.2(c).

What does this operation do to finite-type invariants? Consider an n-circus in the complement of K. After the satellite operation, it becomes an n-circus in the complement of K_L. By the Splitting Relation in Section 3.2, the loops of the double lassos circling K turn into a sum over the components of L at that point.

Let us formalize this.

Definition 4.10. Let X, Y be compact 1-manifolds and $\phi : X \to Y$ a proper map. Let D be a diagram on Y with a fixed parameterization of Y, so that the legs of D have a well-defined position in Y, which are generic with respect to critical points of ϕ. A *lift* of D by ϕ is a diagram D' on X with the same internal part on X so that the image of each leg of D' under ϕ is the corresponding leg of D. Each lift comes with an orientation induced from the local map.

Definition 4.11. Let X, Y be compact 1-manifolds and $\phi : X \to Y$ a proper map. The *pullback* $\phi^* : \mathcal{A}(Y) \to \mathcal{A}(X)$ on a diagram $D \in \mathcal{A}(Y)$ is the sum over all lifts of D from Y to X.

ϕ^* is well-defined (i.e., does not depend on the position of the legs of D and descends modulo the relation) and is invariant under homotopy of ϕ.

Proposition 4.12 (Kuperberg [14]). *Let X, Y be 1-manifolds, let $\eta : Y \to N(X)$ be a pattern for satelliting, and let $\iota : X \to \mathbb{R}^3$ be a link/tangle with an n-circus C in its complement. Then for any universal invariant Z, $Z(\delta_C(X_\eta)) = \phi^*(\mathrm{Diag}(C)) +$ h.o.t, where ϕ is the composition of η and the retraction $N(X) \to X$.*

Proof. By the definition of a universal finite type invariant Z,

$$Z(\delta_C(X_\eta)) = \mathrm{Diag}(C_\eta) + \text{h.o.t.},$$

where C_η is the n-circus C in the complement of X_η. By the Splitting relation, this is $\phi^*(\mathrm{Diag}(C))$. $\square$

Corollary 4.13. *The map $\eta^* : \mathcal{A}(X) \to \mathcal{A}(Y)$ is ϕ^* plus higher order terms (terms that increase the degree of a diagram).*

The nice property of the two cabling operations is that the above formula becomes exact: the "higher order terms" vanish.

Definition 4.14. The operations

$$\Delta^x_{x_1\dots x_n} : \mathcal{A}(|_x \cup X) \to \mathcal{A}(|_{x_1} \cup \cdots \cup |_{x_n} \cup X)$$

or

$$\Delta^x_{x_1\dots x_n} : \mathcal{A}(\bigcirc_x \cup X) \to \mathcal{A}(\bigcirc_{x_1} \cup \cdots \cup \bigcirc_{x_n} \cup X)$$

are the pullback of the n-fold disconnected cover of the component labelled x. When we do not care about the labels on the result, an alternate notation is $\Delta^{(n)}_x$.

Proposition 4.15 (Le and Murakami [16]). *Let L be a framed tangle and $\mathrm{DCable}^n_x(L)$ be the n-fold disconnected cable of L along a knot component x or interval component with one upper and one lower boundary. Then*

$$Z(\mathrm{DCable}^n_x(L)) = \Delta^{(n)}_x(Z(L)).$$

Proof in Section 4.5.2.

Definition 4.16. The operation

$$\psi^{(n)}_x : \mathcal{A}(\bigcirc_x \cup X) \to \mathcal{A}(\bigcirc_x \cup X)$$

is the pullback of the n-fold connected cover of the circle labelled x.

Proposition 4.17 (T. Le). *Let L be a framed tangle and $\mathrm{CCable}^n_x(L)$ be the n-fold connected cable of L along a knot component x as in Figure 4.2(b). Then*

$$Z(\mathrm{CCable}^n_x(L)) = \psi^{(n)}_x(Z(L) \#_x \exp(\frac{1}{2n} \frown))$$

for an appropriate choice of framing on $\mathrm{CCable}^n_x(L)$.

Proof in Section 4.5.3.

One reason to introduce symmetrized diagrams is that the operations Δ and ψ above become very simple in $\mathcal{B}$.

Lemma 4.18. *The map*

$$\Delta^x_{x_1\dots x_n} : \mathcal{A}(*_x \cup X) \to \mathcal{A}(*_{x_1} \cup \cdots \cup *_{x_n} \cup X)$$

is the sum over all ways of replacing each x leg by one of the x_i. $\square$

Remark 4.19. Δ is similar to a coassociative, cocommutative coproduct in a coalgebra, except that it does not take values in $\mathcal{A} \otimes \mathcal{A}$. Do not confuse it with the coproduct $\square$ of Definition 4.3, which is an honest coproduct.

The operation Δ in Lemma 4.18 is analogous to a change of variables $x \mapsto x_1 + \cdots + x_n$ for ordinary functions $f(x)$. We will use a suggestive notation: a leg labelled by a linear combination of variables means the sum over all ways of picking a variable from the linear combination. If $D(x)$ is a diagram with some legs labelled x, $\Delta^{(n)}(D(x)) = D(x_1 + \cdots + x_n)$ is the diagram with the same legs labelled $x_1 + \cdots + x_n$.

Lemma 4.20. *The map*

$$\psi^{(n)}_x : \mathcal{A}(\circledast_x \cup X) \to \mathcal{A}(\circledast_x \cup X)$$

is multiplication by n^k on diagrams with k legs labelled x. $\square$

This operation is related to the change of variables $x \mapsto nx$.

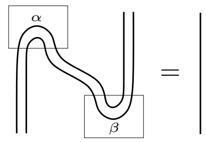


Figure 4.3: A doubled wiggle.

4.5.2 Disconnected cabling

Here we sketch the proof of Proposition 4.15. For full details, please see Le and Murakami [16].

First consider the disconnected cabling of a braid. Take one strand of a braid and replace it by a number of copies parallel with a small displacement ϵ. Because the strands are parallel, there are no new chords between them ($z_1 - z_2$ is constant); since the new strands are close to the original strand, thep integrals for chords between the doubled strands and another strand are close to chords for the original strand, with equality in the limit as $\epsilon \to 0$. See Lescop [17] for the analytic details. Hence for parallel cabling of a braid Proposition 4.15 holds.

When we switch to tangles, there may be cups and caps. The Kontsevich integral of a doubled maximum $Z(\Cap)$ will not be the double of the Kontsevich integral of a single maximum $Z(\cap)$.

Let $\alpha = \lim_{\epsilon\to 0} Z(\Cap_\epsilon)$ where $\Cap_\epsilon$ is a doubled maximum with the distances between the strands at the bottom ϵ, 1, and ϵ in order. (There are naively two logarithmic divergences in this integral which fortuitously cancel.) Note that α includes renormalization of ν on each strand added so that $Z(\cap\!\cup) = Z(|)$. Similarly we get another element $\lim_{\epsilon\to 0} Z(\Cup_\epsilon) = \beta \in \mathcal{A}(\uparrow\uparrow)$ at minima. Although we can say little about α or β by themselves,[1] we can compute their product. Consider a doubled wiggle, as in Figure 3. It can be decomposed into three pieces: a doubled maximum, a doubled minimum, and the doubling of the braid whose closure is $\cap\!\cup$. The doubled ends of the Kontsevich integral of the doubling of the braid can slide over any diagram which, like α or β, lives on two strands, so we can collect α and β at the beginning. We then recognize the remaining integral as the double of the naive integral which gave us ν^{-1} above. Since this whole $(2, 2)$ tangle can be straightened to give the identity, we find

$$\alpha \cdot \beta \cdot \Delta\nu^{-1} = 1$$

or

$$\alpha \cdot \beta = \Delta\nu$$

where the multiplication is the natural multiplication on $\mathcal{A}(\uparrow\uparrow)$.

Now consider doubling a knot component or a component with one upper and one lower end. There will be the same number k of maxima and minima, so the net error from a pure doubling will be some product of k α's and k β's. As before, we can slide the β's through the rest of the integral until they are next to the α's; then each pair cancel to give $\Delta\nu$, which is exactly the double of the original renormalization, as desired. (There is one difficulty we are brushing under the rug: there are actually two versions of α and β, depending on which direction you traverse the maximum, and several different products can occur. To show that all pairs cancel, some more complicated version of Figure 3 need to be considered.)

So Proposition 4.15 is true for doubling a knot component. By iterating this argument, the same thing is true for an n-fold disconnected satellite. $\square$

[1] Le and Murakami [16] have analyzed contexts in which you can compute α and β and, in particular, when α and β are the doubles of a maximum and minimum.

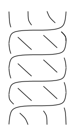
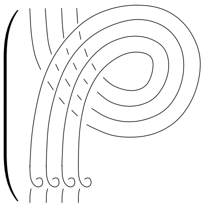
4.5.3 Connected cabling

Now we consider the case of a connected cabling. The difference between the connected cabling and the disconnected cabling above is the extra $1/n$ twist T_n inserted at one point:

$$T_n =$$  .

By isotopy can assume that this twist occurs in a horizontal slice where all the other strands are vertical. We can apply Proposition 4.15 on the (n, n) "tangle" obtained by excising T_n. (This object is not properly a tangle, since there is a little piece cut out of it. But we can still compute its Kontsevich integral by Proposition 4.15.) To complete the computation, we need to compute $a := Z(T_n)$.

Repeating T_n n times, we get a full twist which we can compute using Lemma 4.8 and the disconnected cabling of the previous section:

$Z\Bigg($  $\Bigg) = Z\Bigg($  $\Bigg)$

$$= \Delta^{(n)}\left(\exp(\frac{1}{2}\frown)\right)\exp(-\frac{1}{2}\frown)^{\otimes n} =: b.$$

The notation $\exp(-\frac{1}{2}\frown)^{\otimes n}$ means n copies of the framing change element $\exp(-\frac{1}{2}\frown)$, one on on each of the n strands.

The n copies of T_n that appear are not quite the same: they differ by cyclic permutations of the strands. If we arrange the n strands at the top and bottom of T_n to be at the vertices of a regular n-gon, the strands are symmetric and $a^n = b$ or

$$a = b^{\frac{1}{n}} = \Delta^{(n)}\left(\exp(\frac{1}{2n}\frown)\right)\exp(-\frac{1}{2n}\frown)^{\otimes n}.$$

More generally, we will need to conjugate T_n by some element C to get the strands symmetric; this implies that

$$a = cb^{\frac{1}{n}}S(c^{-1})$$

where $c = Z(C)$ and S is the automorphism of $\mathcal{A}(\uparrow_{x_1} \ldots \uparrow_{x_n})$ which rotates the strands by $x_i \mapsto x_{i-1}$.

Lemma 4.21. *$Z(T_n) = c\Delta^{(n)}\left(\exp(\frac{1}{2n}\frown)\right)\exp(-\frac{1}{2n}\frown)^{\otimes n}S(c^{-1})$ for some $c \in \mathcal{A}(\uparrow_{x_1} \ldots \uparrow_{x_n})$.* □

This lemma can also be proved without using the specific geometry of the Kontsevich integral in an argument due to D. Bar-Natan, but we will not give the argument here.

Proof of Proposition 4.17. By the above computations, the invariant of the connected cable of a knot K is $\Delta^{(n)}(Z(K))$, multiplied by $Z(T_n)$, and closed up with a twist. The conjugating elements c and $S(c^{-1})$ can be swept through the knot and cancel each other. The factor $\Delta^{(n)}(\exp(\frown/2n))$ in a can be combined with $Z(K)$ so that we apply $\Delta^{(n)}$ to $Z(K)\#\exp(\frown/2n)$. The twisted closure turns $\Delta^{(n)}$ into $\psi^{(n)}$. The remaining n factors of $\exp(-\frown/2n)$ in a can be slid around the knot and combined to give

$$Z(\mathrm{CCable}^n(K)) = \psi^{(n)}\left(Z(K)\,\#\,\exp(\frac{1}{2n}\frown)\right)\#\,\exp(-\frac{1}{2}\frown).$$

The last factor is absorbed in the "appropriate choice of framing" in the statement of the proposition. □

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Chapter 5

Wheeling

Now that we have reviewed the basic theory of Vassiliev invariants, the proof of Theorem 2 is relatively straightforward. In Section 5.1 we will show how to interpret the Hopf link ϕ of Figure 1.1(a) as a map $\Phi : \mathcal{B} \to \mathcal{A}$ by taking its Kontsevich integral in $\mathcal{A}(|\circledast)$ and gluing the legs on the bead to the diagram in $\mathcal{B}$. In Section 5.2 we will see how the equation "$1 + 1 = 2$" of Figure 1.1(b) implies that Φ is multiplicative. Briefly, we interpret both sides of the equation as maps from $\mathcal{B} \otimes \mathcal{B}$ to $\mathcal{A}$. The "$1 + 1$" side takes (X, Y) to $\Phi(X) \# \Phi(Y)$ and the "2" side takes (X, Y) to $\Phi(X \mathbin{\dot\cup} Y)$. The fact that the two links are isotopic implies that the two maps are equal. In Section 5.3 we relate this map Φ to the wheeling map Υ of Section 2.7 by showing that the lowest degree term Φ_0 of Φ is also multiplicative and is the same as the Duflo map with possibly different coefficients of the wheels ω_n. To fix the coefficients, it suffices to do a small computation for some Lie algebra; we do the computation for $\mathfrak{sl}_2$ in Appendix B.

5.1 The map Φ

We start by defining a kind of inner product on the space $\mathcal{B}$.

Definition 5.1. For a diagrams $C, C' \in \mathcal{B}$ so that C has no struts, the *inner product* of C and C' is

$$\mathcal{A}(\emptyset) \ni \langle C, C' \rangle = \begin{cases} \text{the sum of all ways of gluing all the legs of } C \text{ to all legs of } D & \text{if } C \text{ and } C' \text{ have the same number of legs} \\ 0 & \text{otherwise} \end{cases}$$

We will sometimes want to fix C and consider $\langle C\,, \cdot\rangle$ as a map from $\mathcal{B}$ to $\mathcal{A}(\emptyset)$; we will denote this map $\iota(C)$. This definition works equally well in the presence of other skeleton components or to glue several components. We will use subscripts to indicate which ends are glued.

As in Definition 2.8, the restriction that C not have struts is to guarantee convergence and avoid closed circles.

There are two dualities relating $\langle \cdot\,, \cdot \rangle$ with other operations we have defined.

Lemma 5.2. *Multiplication and comultiplication in $\mathcal{B}$ are dual in the sense that*

$$\langle C, D_1 \mathbin{\dot\cup} D_2 \rangle = \langle \Delta_{xy} C, (D_1)_x \otimes (D_2)_y \rangle_{xy}.$$

Similar statements hold in the presence of other ends.

Proof. The glued diagrams are the same on the two sides; we either combine the legs of D_1 and D_2 into one set and then glue with C, or we split the legs of C into two pieces which are then glued with D_1 and D_2. (Note that there are no combinatorial factors to worry about: in both cases, we take the sum over all possibilities.) ∎

Lemma 5.3. *Multiplication by a diagram $B \in \mathcal{B}$ and applying B as a diagrammatic differential operator are adjoint in the sense that*

$$\langle A \cup B, C \rangle = \langle A, \partial_B(C) \rangle.$$

Proof. As before, the diagrams are the same on both sides. □

The map Φ of the paper is constructed from the bead on a wire in Figure 1.1(a). We start with its Kontsevich integral:

$$Z(\phi^z_x) \in \mathcal{A}(\uparrow_z, \circlearrowright_x).$$

We then symmetrize on legs attached to the bead x as explained in Section 3.3:

$$\chi_x^{-1} Z(\phi^z_x) \in \mathcal{A}(\uparrow_z, \circledast_x).$$

Finally, we use the inner product operation along the legs x to get a map from $\mathcal{B}$ to $\mathcal{A}$:

$$\Phi = \iota_x \chi_x^{-1} Z(\phi^z_x) : \mathcal{B} \to \mathcal{A}$$

In this last step, there are two things we have to check. First, we must see that $\chi^{-1} Z(\phi)$ has no struts. By Lemma 4.9, this follows from the fact that we took the bead with the zero framing. Second, we need to check that the inner product descends modulo the link relations on x in $\mathcal{A}(\uparrow_z, \circledast_x)$.

Lemma 5.4. *The inner product $\langle \cdot\, , \cdot \rangle_x : \mathcal{A}(*_x \cup X) \otimes \mathcal{A}(*_x) \to \mathcal{A}(X)$ descends to a map $\langle \cdot\, , \cdot \rangle_x : \mathcal{A}(\circledast_x \cup X) \otimes \mathcal{A}(*_x) \to \mathcal{A}(X)$.*

Proof. We use a sliding argument similar to the one in the proof of Lemma 3.17. Link relations in $\mathcal{A}(\circledast_x \cup X)$ can be slid over diagrams in $\mathcal{A}(*_x)$, as shown in Figure 5.1. □

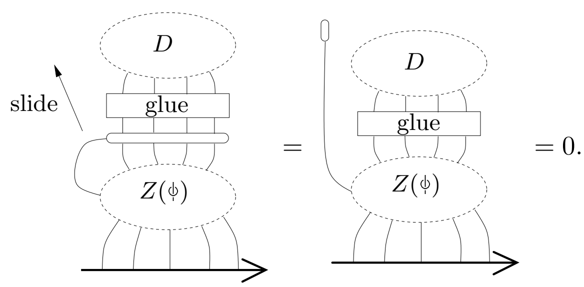


Figure 5.1: The proof that $\Phi(D)$ is well-defined modulo link relations on $Z(\phi)$: link relations in $Z(\phi)$ can be slid over D.

5.2 Multiplicativity of Φ

We now come to the key lemma in the proof of Theorem 2.

Lemma 5.5. *The map $\Phi : \mathcal{B} \to \mathcal{A}$ is an algebra map.*

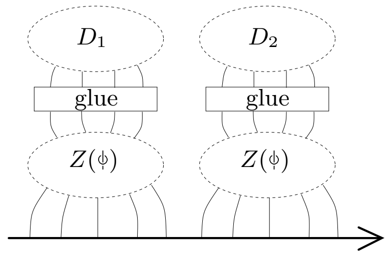


Figure 5.2: Gluing $Z(\phi \# \phi)$ to $D_1 \otimes D_2$.

Proof. As advertised, we use the equality of links "$1+1=2$". Let us see what this equality of links says about the Kontsevich integral of the Hopf link. On the "$1+1$" side, we see the connected sum of two open Hopf links; by Section 4.4, the invariant of the connected sum is the connected sum of the invariants. To write this conveniently, let $H(z;x)$ be $Z(\phi) \in \mathcal{A}(\uparrow_z, \circledast_x)$, with the wire labelled by z and the bead labelled by x. Then

$$Z(\phi \# \phi) = H(z;x_1) \#_z H(z;x_2) \in \mathcal{A}(\uparrow_z, \circledast_{x_1}, \circledast_{x_2}).$$

On the "2" side, we see the disconnected cable of a Hopf link. By Section 4.5.2, this becomes the coproduct Δ:

$$Z(\mathrm{DCable}(\phi)) = \Delta^x_{x_1x_2} H(z;x). \in \mathcal{A}(\uparrow_z, \circledast_{x_1}, \circledast_{x_2}).$$

Since the two tangles are isotopic, we have

$$H(z;x_1,x_2) \stackrel{\mathrm{def}}{=} H(z;x_1) \#_z H(z;x_2) = \Delta^x_{x_1x_2} H(z;x) \in \mathcal{A}(\uparrow_z, \circledast_{x_1}, \circledast_{x_2}).$$

Now consider the map

$$\Xi = \iota_{x_1}\iota_{x_2} H(z;x_1,x_2) : \mathcal{B} \otimes \mathcal{B} \to \mathcal{A};$$

in other words, in $\Xi(D_1 \otimes D_2)$ glue the x_1 and x_2 legs of $H(z;x_1,x_2)$ to D_1 and D_2 respectively. This descends modulo the two different link relations in $\mathcal{A}(\uparrow, \circledast, \circledast)$ by the argument of Figure 5.1, applied to D_1 and D_2 separately. We have two different expressions for this map from the two different expressions for $H(z;x_1,x_2)$. On the "$1+1$" side, the gluing does not interact with the connected sum and we have

$$\Xi(D_1, D_2) = \Phi(D_1) \# \Phi(D_2).$$

See Figure 5.2. For the "2" side, we use Lemma 5.2 to see that

$$\begin{aligned} \Xi(D_1, D_2) &= \langle \Delta^x H(z;x), D_1 \otimes D_2 \rangle && (5.1)\\ &= \langle H(z;x), D_1 \mathbin{\dot\cup} D_2 \rangle && (5.2)\\ &= \Phi(D_1 \mathbin{\dot\cup} D_2). && (5.3) \end{aligned}$$

See Figure 5.3.

Combining the two, we find

$$\Phi(D_1) \# \Phi(D_2) = \Phi(D_1 \mathbin{\dot\cup} D_2).$$

□

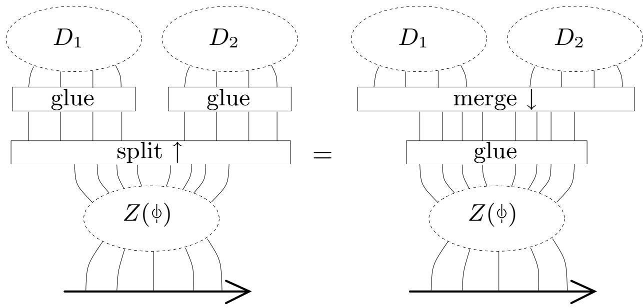


Figure 5.3: Gluing $Z(\mathrm{DCable}\,\phi)$ to $D_1 \otimes D_2$ in two equivalent ways.

5.3 Mapping degrees and the Duflo isomorphism

We have successfully constructed a multiplicative map from $\mathcal{B}$ to $\mathcal{A}$. We will see later (see Section 6.4) that this map Φ is the Duflo map, but we cannot yet see this. Instead we will consider the lowest degree term Φ_0 of Φ.

Definition 5.6. The *mapping degree* of a diagram $D \in \mathcal{A}(\uparrow_z, \circledast_x)$ with respect to x is the amount $\iota_x D : \mathcal{B} \to \mathcal{A}$ shifts the degree. Explicitly, it is the degree of D minus the number of x legs of D.

Since there are no x-x struts in $H(z;x)$, every x leg of H must be attached to another vertex (either internal or on the interval z). Furthermore, if two x legs are attached to the same interal vertex, the diagram vanishes by antisymmetry. Therefore there are at least as many other vertices as x legs in H and the mapping degree is ≥ 0.

Definition 5.7. $H_0(z;x)$ is the part of $H(z;x)$ of mapping degree 0 with respect to x. Φ_0 is $\iota_x H_0(z;x)$.

Φ_0 is still multiplicative, since the multiplications in $\mathcal{A}$ and $\mathcal{B}$ both preserve degrees. (For homogeneous diagrams D_1 and D_2 of degrees n_1 and n_2, $\Phi_0(D_1 \cup D_2)$ is the piece of $\Phi(D_1 \cup D_2)$ of degree $n_1 + n_2$ and likewise for $\Phi_0(D_1) \# \Phi_0(D_2)$.)

The diagrams that appear in H_0 are very restricted, since every vertex that is not an x leg must connect to an x leg; since these vertices are trivalent, the other two incident edges form a 1-manifold. The possible diagrams are x wheels and $x - z$ struts, as shown in Figure 5.3. The linking number between the bead and the wire in the link ϕ is 1, so the coefficient of the x-z strut is 1. Combined with the fact that the Kontsevich integral is grouplike, we find that

$$H_0(z;x) = \exp(\overset{x}{\underset{z}{|}}) \cup \exp(\Omega')$$
$$\Omega' = \sum_n a_{2n}\omega_{2n}$$

for some coefficients a_n. Note that the right hand side is written in $\mathcal{A}(\uparrow, *)$ (with a strange mixed product), since there is no algebra structure on $\mathcal{A}(\uparrow, \circledast)$.

By the following lemma, we now have a multiplicative map very similar to our desired map Υ.

Lemma 5.8. $\Phi_0 = \chi \circ \partial'_\Omega$.

Proof. Using Lemma 5.3 and noting that gluing with $\exp(|^x_z)$ takes the legs of a diagram in $\mathcal{B}$ and averages over all ways of ordering them, as in the definition of χ, we see that

$$\Phi_0(D) = \langle \exp(\overset{x}{\underset{z}{|}} \cup \Omega', D\rangle = \langle \exp(\overset{x}{\underset{z}{|}}, \partial'_\Omega(D)\rangle = \chi(\partial'_\Omega(D)).$$

□

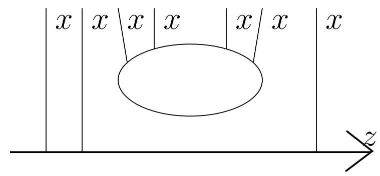


Figure 5.4: The only diagrams in $\mathcal{A}(\uparrow_z, \circledast_x)$ of mapping degree 0 with respect to x are wheels and struts.

To see that $\Phi_0 = \Upsilon$, we only need to check that $\Omega = \Omega'$. This is proved in Proposition B.4.

Chapter 6

Wheels and the Kontsevich integral of the unknot

6.1 Results

In this chapter, we will give some explicit computations of the Kontsevich integral to all orders for some specific knots and links. We begin by reviewing our principal results. The first one was already stated in the introduction.

Theorem 3 (Wheels; joint with T. Le). *The Kontsevich integral of the unknot is*

$$Z(\bigcirc) = \nu = \Omega \in \mathcal{B}.$$

Recall that Ω is the "wheels" element from the introduction.

By changing the framing on the unknot and cabling it, we can construct a Hopf link. Using the results of Section 4.5.3 and the value of $Z(\bigcirc)$, we can compute the invariant of the Hopf link from the invariant of the unknot. There are several good formulas for the answer.

Theorem 4. *The framed Kontsevich integral of the Hopf link can be expressed in the following equivalent ways:*

$$Z({}_x\bigcirc\!\!\!\bigcirc_y) = \begin{cases} \Upsilon_x \circ \Upsilon_y(\exp({}^{y}\!\frown^{x})) \cdot (\mathrm{Vacuum}) \\ \Upsilon_x(\exp_{\cup}({}^{y}\!\frown^{x})\Omega_x) \cdot (\mathrm{Vacuum}) \end{cases}$$
$$Z(\phi^y_x) = \exp({}^{y}\!\frown^{x}) \cup \Omega_y,$$

for some elements $(\mathrm{Vacuum}) \in \mathcal{A}(\emptyset)$.

In the last expression, ϕ^y_x is the $(1,1)$ tangle whose closure is the Hopf link, with the bead labelled by y and the wire labelled by x. From this last equality in Theorem 4, we can see exactly the map Φ from Chapter 5.

Corollary 6.1. $\Phi = \Phi_0 = \chi \circ \partial_\Omega$.

6.2 Useful facts

The element $\Omega \in \mathcal{B}$ is a very remarkable element. We will need one of its nice properties for the proof of the Wheels theorem. Start from the basic equality proved in the Wheeling theorem,

$$\Delta^x_{x_1 x_2} H_0(z;x) = H_0(z;x_1) \#_z H_0(z;x_2) \in \mathcal{A}(\uparrow_z, \circledast_{x_1}, \circledast_{x_2}) \quad \text{where} \quad H_0(z;x) = \Omega_x \exp({}^{x}\!\frown^{z}).$$

Now consider dropping the strand z, i.e., mapping all diagrams with a z vertex to 0. (Knot-theoretically, this corresponds to dropping the central strand in the equation "$1+1=2$".) We find

$$\Delta\Omega = \Omega \otimes \Omega \in \mathcal{A}(\circledast\circledast). \tag{6.1}$$

Note that this equality is not true inside $\mathcal{A}(**)$.

Lemma 6.2 (Pseudo-linearity of $\log\Omega$). *For any $D \in \mathcal{B}$,*

$$\partial_D(\Omega) = \partial_D\Omega|_0 \cup \Omega = \langle D, \Omega\rangle\Omega.$$

Proof.

$$\partial_D(\Omega)_x = \langle D_y, \Omega_{x+y}\rangle_y = \langle D_y, \Omega_x\Omega_y\rangle_y = \langle D_y, \Omega_y\rangle_y\Omega_x.$$

In the second equality, we use Equation 6.1. This is allowed, since the contraction descends to $\mathcal{A}(\circledast\circledast) \simeq \mathcal{A}(\circledast\uparrow)$ by the argument of Lemma 5.4. Note that it is crucial that D is invariant for this argument. ☐

Remark 6.3. Compare this lemma with standard calculus: if D is any differential operator and f is a linear function, then $De^f = (Df)(0)e^f$. The prefix "pseudo" is written above because Lemma 6.2 does not hold for every D, but only for x-invariant D's. (E.g., if D were in $\mathcal{A}(**)$ rather than $\mathcal{A}(*)$, the lemma would not be true.)

Although we are interested in knots and links in S^3 in this thesis, for which the appropriate space of diagrams $\mathcal{A}^{\mathrm{bc}}$ from Section 3.2 is boundary connected, vacuum diagrams (elements of $\mathcal{A}(\emptyset)$) appear at various points. Notably, the wheeling map Υ does not preserve the subspace of boundary connected diagrams. Although the resulting vacuum components can be computed explicitly,[1] they are almost always irrelevant for us and it would just complicate the formulas to keep track of them. To avoid this, we will introduce the *boundary-connected projection* $\pi^{\mathrm{bc}} : \mathcal{A} \to \mathcal{A}^{\mathrm{bc}}$ which maps any diagram containing vacuum components to 0 and is otherwise the identity. Note that π^{bc} is multiplicative. There are similar projections, which we also be called π^{bc}, for other spaces $\mathcal{A}(X)$.

If we compose Lemma 6.2 with π^{bc}, we find

$$\pi^{\mathrm{bc}}\partial_D\Omega = \begin{cases} \Omega & D \text{ is the empty diagram} \\ 0 & \text{otherwise} \end{cases}. \tag{6.2}$$

6.3 Coiling the unknot

The basic equation we will use to identify ν is "$n \cdot 0 = 0$" from the introduction: the n-fold connected cable of the unknot is the unknot with a new framing. From Section 4.5.3, this implies that

$$\psi^{(n)}(\nu \# \exp_{\#}(\frac{1}{2n}\frown)) = \nu \# \exp_{\#}(\frac{n}{2}\frown). \tag{6.3}$$

This equation is true for all $n \in \mathbb{Z}$, $n > 0$. In each degree, each side is a Laurent polynomial in n of bounded degree; therefore, the two sides are equal as Laurent polynomials. The RHS is a polynomial in n, so both sides are polynomials (i.e., have no negative powers of n.) Let us evaluate both sides at $n = 0$. On the RHS, we get just ν. For the LHS, recall how $\psi^{(n)}$ acts in the space $\mathcal{B}$: it multiplies a diagram with k legs by n^k. Each strut appearing in the product on the LHS contributes a factor of $1/n$; each leg in the symmetrized result gives a factor of n. To bound how many legs can appear in a symmetrized product, we use the following lemma.

Lemma 6.4. *For any elements $x_1, \ldots, x_k \in \mathcal{A}(\uparrow)$ with at least one leg on the interval $\uparrow$, $\chi^{-1}(x_1 \# \cdots \# x_k) \in \mathcal{B}$ has at least k legs.*

[1] D. Bar-Natan and R. Lawrence [5] have done these computations

Proof. First note that any vacuum diagrams that appear in the x_i's pass through unchanged to the result; let us assume that there are none, so that we can use the vacuum projection π^{bc} without changing the result. By Theorem 2,

$$\pi^{\mathrm{bc}}\chi^{-1}(x_1 \# \ldots \# x_n) = \pi^{\mathrm{bc}}\partial_\Omega(\Upsilon^{-1}(x_1) \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \Upsilon^{-1}(x_k)).$$

Let $y_i = \pi^{\mathrm{bc}}\Upsilon^{-1}(x_i)$. Each y_i has at least one leg, since if the ∂_Ω^{-1} of Υ^{-1} eats all the legs of $\chi^{-1}x_i$, it also creates a vacuum diagram which is killed by π^{bc}. Then

$$\pi^{\mathrm{bc}}\partial_\Omega(y_1 \ldots y_k) = \pi^{\mathrm{bc}}\langle \Omega_a, \Delta_{ab}(y_1 \ldots y_k)\rangle_a.$$

Let $\Delta_{ab}y_i = (y_i)_a + z_i$; diagrams in z_1 have at least one b leg. We see that

$$\begin{aligned}
\pi^{\mathrm{bc}}\langle \Omega_a, (y_1)_a \Delta_{ab}(y_2 \ldots y_n)\rangle_a &= \pi^{\mathrm{bc}}\langle (\partial_{y_1}\Omega)_a, \Delta_{ab}(y_2 \ldots y_k)\rangle_a && \text{by Lemma 5.3}\\
&= 0. && \text{by Equation 6.2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi^{\mathrm{bc}}\partial_\Omega(y_1 \ldots y_k) &= \pi^{\mathrm{bc}}(\langle \Omega_a, (y_1)_a \Delta_{ab}(y_2 \ldots y_k)\rangle_a + \langle \Omega_a, z_1 \Delta_{ab}(y_2 \ldots y_k)\rangle_a)\\
&= \pi^{\mathrm{bc}}\langle \Omega_a, z_1 \Delta_{ab}(y_2 \ldots y_k)\rangle_a\\
&= \cdots\\
&= \pi^{\mathrm{bc}}\langle \Omega_a, z_1 \ldots z_k\rangle_a.
\end{aligned}$$

Each z_i has at least one leg labelled b, so the product has at least k legs labelled b which are the legs in the result. □

Consider expanding the exponential $\exp_\#(\frown/2n)$ in the LHS of Equation 6.3. In the term with k struts, there is a factor of $1/n^k$ from the factors in front of the struts. On the other hand, by Lemma 6.4, the product has at least k legs, or $k+1$ if there is a non-trivial contribution from ν. Since the overall power of n is $n^{k-\#\text{ legs}}$, when we evaluate at $n=0$ the term ν does not contribute at all.

$$\psi^{(n)}(\nu \# \exp_\#(\frac{1}{2n}\frown))|_{n=0} = \psi^{(n)}(\exp_\#(\frac{1}{2n}\frown))|_{n=0}.$$

Now we want to pick out the term from $(\frown)^{\#k}$ with exactly k legs. We can do this computation explicitly using the wheeling map Υ. Alternatively, the result (which is ν) must be a diagram of degree k and with k legs, hence of mapping degree 0: $\nu = \nu_0$. $\Omega \cap |$ was shown in Chapter 5 the be the part of $Z(\phi)$ of mapping degree 0. Dropping the central strand from ϕ leaves an unknot, so $\Omega = \nu_0 = \nu$. This completes the proof of Theorem 3. □

Exercise 6.5. Do the computation suggested above. Show that

$$\exp_\#(\frac{1}{2}\frown) = \Omega \mathbin{\dot\cup} \exp_{\dot\cup}(\frac{1}{2}\frown).$$

Hint 6.6. Use Lemma 6.8.

6.4 From the unknot to the Hopf link

We will now apply Theorems 2 and 3 to compute the invariant of the Hopf link. A little attention is required in order to perform the operations in the correct order. We start by computing the Kontsevich integral of the $+1$ framed unknot.

$$\begin{aligned}
Z(\bigcirc^{+1}) &= \Omega \# \exp_\#(\frac{1}{2}\frown)\\
&= \partial_\Omega\left(\partial_\Omega^{-1}(\Omega) \mathbin{\dot\cup} \exp_{\dot\cup}(\partial_\Omega^{-1}(\frown))\right) && \text{by Theorem 2}\\
&= \pi^{\mathrm{bc}}\partial_\Omega\left(\Omega \mathbin{\dot\cup} \exp(\frown)\right). && \text{by Equation 6.2}
\end{aligned}$$

To pass to the Hopf link, we double $Z(\bigcirc^{+1})$. The following lemma tells us how ∂_Ω intereacts with doubling. We use $\hat{D}$ as an alternate notation for ∂_D so that we can use a subscript to indicate which variable the differential operator acts on.

Lemma 6.7. *For $C, D \in \mathcal{B}$ with C strutless,*

$$\Delta_{xy}\hat{C}(D) = \hat{C}_x(\Delta_{xy}D) = \hat{C}_y(\Delta_{xy}D).$$

□

If we want to apply ∂_Ω^{-1} to both components of the Hopf link, we can compute $\partial_\Omega^{-2}(Z(\bigcirc^{+1}))$. We make an inspired guess.

Lemma 6.8. $\pi^{bc}\partial_\Omega(\exp\frac{1}{2}\frown) = \Omega \cup \exp(\frac{1}{2}\frown)$.

Proof.

$$\begin{aligned}
\pi^{\mathrm{bc}}\partial_\Omega(\exp(\frac{1}{2}\frown)) &= \pi^{\mathrm{bc}}\langle\Omega_y, \exp(\frac{1}{2}\,{}^{x+y}\frown^{x+y})\rangle_y \\
&= \pi^{\mathrm{bc}}\langle\Omega_y, \exp(\frac{1}{2}\,{}^{x}\frown^{x})\exp({}^{y}\frown^{x})\exp(\frac{1}{2}\,{}^{y}\frown^{y})\rangle_y \\
&= \pi^{\mathrm{bc}}\langle\partial_{\exp(\frac{1}{2}\frown)}(\Omega)_y, \exp({}^{x}\frown^{y})\rangle_y \cup \exp(\frac{1}{2}\,{}^{x}\frown^{x}) && \text{by Lemma 5.3} \\
&= \pi^{\mathrm{bc}}\langle\Omega_y, \exp({}^{y}\frown^{x})\rangle_y \cup \exp(\frac{1}{2}\,{}^{x}\frown^{x}) && \text{by Equation 6.2} \\
&= \Omega \cup \exp(\frac{1}{2}\frown).
\end{aligned}$$

□

As a corollary, we see that

$$\pi^{\mathrm{bc}}\partial_\Omega^{-2}(Z(\bigcirc^{+1})) = \exp(\frac{1}{2}\frown). \tag{6.4}$$

We now compute.

$$\begin{aligned}
\pi^{\mathrm{bc}}\Delta_{xy}(\hat{\Omega}^{-2}Z(\bigcirc^{+2})) &= \pi^{\mathrm{bc}}\hat{\Omega}_x^{-1}\hat{\Omega}_y^{-1}Z({}_x^{+1}\text{ⵁ}_y^{+1}) && \text{by Lemma 6.7 and Proposition 4.15} \\
&= \pi^{\mathrm{bc}}\Delta_{xy}(\exp(\frac{1}{2}\frown)) && \text{by Equation 6.4} \\
&= \exp({}^{x}\frown^{y})\exp(\frac{1}{2}\,{}^{x}\frown^{x})\exp(\frac{1}{2}\,{}^{y}\frown^{y}).
\end{aligned}$$

Apply $\Upsilon_x \circ \Upsilon_y$ to both sides. We see that

$$\begin{aligned}
Z({}_x^{+1}\text{ⵁ}_y^{+1}) &= \pi^{\mathrm{bc}}Z({}_x^{+1}\text{ⵁ}_y^{+1}) \\
&= \pi^{\mathrm{bc}}\Upsilon_x \circ \Upsilon_y(\exp({}^{x}\frown^{y}) \cup \exp(\frac{1}{2}\,{}^{x}\frown^{x}) \cup \exp(\frac{1}{2}\,{}^{y}\frown^{y})) \\
&= \pi^{\mathrm{bc}}\Upsilon_x \circ \Upsilon_y(\exp({}^{x}\frown^{y})) \# \exp_\#(\frac{1}{2}\ominus_x) \# \exp_\#(\frac{1}{2}\ominus_y)
\end{aligned}$$

so

$$Z({}_x\text{ⵁ}_y) = \pi^{\mathrm{bc}}\Upsilon_x \circ \Upsilon_y(\exp({}^{x}\frown^{y})).$$

This is the first equality of Theorem 4. For the second equality,

$$\hat{\Omega}_y(\exp({}^{x}\frown^{y})) = \Omega_x \cup \exp({}^{x}\frown^{y}).$$

so

$$Z({}_x\bigcirc\!\!\bigcirc_y) = \pi^{\mathrm{bc}}\Upsilon_x(\exp({}^{y}\frown{}^{x})\Omega_x).$$

For the last equality of the theorem, multipicativity of Υ implies that

$$\begin{aligned}\pi^{\mathrm{bc}}\Upsilon_x(\exp({}^{y}\frown{}^{x})\Omega_x) &= \pi^{\mathrm{bc}}(\Upsilon_x(\exp({}^{y}\frown{}^{x}))\,\#\,\Upsilon_x(\Omega_x)) \\ &= \pi^{\mathrm{bc}}(\Upsilon_x(\exp({}^{y}\frown{}^{x})))\,\#\,\chi(\Omega_x) \qquad = \chi(\exp({}^{y}\frown{}^{x})\,\dot\cup\,\Omega_y)\,\#\,\chi(\Omega_x).\end{aligned}$$

But by Equation 4.3,

$$Z(\phi_x^y) = Z({}_x\bigcirc\!\!\bigcirc_y)\,\#\,\Omega_x^{-1} = \exp({}^{y}\frown{}^{x})\,\dot\cup\,\Omega_y.$$

This completes the proof of Theorem 4. $\square$

Appendix A

Cyclic invariance of vertices

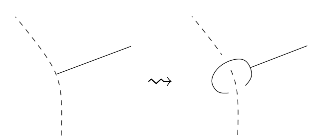
In this appendix, we tie up a loose end from Section 3.2. In that section, the diagrams we found naturally from the finite-type theory were Jacobi diagrams with an additional structure, a "routing". Recall that a routing of a Jacobi diagram is a choice of two edges incident to each internal trivalent verex.

Here we will find some good routings (routings that corresponds to n-circuses) for any boundary-connected Jacobi diagram and show that all such routings are equal modulo the Antisymmetry and Splitting relations.

First we will give a convenient criterion for a routing to be good in the above sense.

Lemma A.1. *A routing of a Jacobi diagram D is good if there exists an ordering of the internal vertices of D so that every vertex has at least one neighbor along a distinguished edge which is either an external vertex or comes earlier in the ordering.*

Proof. We need to check that after resolving each vertex like



we are left with a collection of n double lassos that are trivial in S^3 once you forget the knot. The internal vertices can be pulled apart one by one in the specified order: at each stage, at least one side of the encircled lasso ends in a loop with nothing inside. □

In fact, the condition in the lemma is an if and only if, but we do not need that.

Definition A.2. With respect to an ordering of the internal vertices of a Jacobi diagram, a vertex v_1 is *younger* than a vertex v_2 if v_1 is an internal vertex or v_1 and v_2 are both internal and v_1 is earlier than v_2 in the ordering. An ordering of the internal vertices is *good* if every vertex has a younger neighbor. A routing and an ordering of the internal vertices are *compatible* if they satisfy the condition of Lemma A.1, i.e., if one of the two distinguished neighbors of each vertex is younger.

Proposition A.3. *Every boundary connected Jacobi diagram D has at least one good routing.*

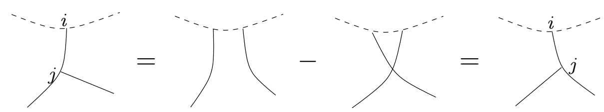
Proof. Because D is boundary connected, there is a good ordering on the internal vertices. (Order the vertices from the external vertices on the knot inward.) Every good ordering has a comptabile routing. □

Lemma A.4. *Any two good orderings are related by a series of transpositions of vertices adjacent in the ordering.*

Proof. The minimal vertex in the first ordering must have an external vertex as a neighbor. Therefore, there is no obstruction to moving this vertex to the first position in the second ordering by a series of transpositions. We can repeat this for each vertex in turn. □

Lemma A.5. *Any two two routings compatible with the same ordering are equivalent modulo the vertex and antisymmetry relations.*

Proof. We need to check that the two possibilities for the routing at each internal vertex are equivalent. This follows from two applications of the vertex relation:



where i is younger than j and the order of the two vertices in the middle sum is chosen depending on which neighbor of i is younger, or is irrelevant if i is an external vertex. □

Proposition A.6. *All routings of a Jacobi diagram D compatible with any ordering are equal modulo the antisymmetry and vertex relations.*

Proof. Pick a good ordering for each routing. We can adjust the routing while keeping the ordering fixed by Lemma A.5, so we just need to check that we can change the ordering. The two orderings are related by a chain of good orderings related by adjacent transpositions by Lemma A.4. At each transposition, the two vertices involved must have younger neighbors which are not each other (otherwise we could not exchange them); a routing in which these two neighbors are on distinguished edges is compatible with both orderings. □

Remark A.7. Goussarov [9] and Habiro [11] independently found a topological theory of Jacobi diagrams in which the vertices are cyclically invariant from the beginning.

Appendix B

Fixing the coefficients in Ω

To fix the coefficients in Ω (in terms of Chapter 5, to show that $\Omega' = \Omega$), we look at

$$\partial_{\Omega'}(\exp(\frac{1}{2}\frown))$$

as we did in Chapter 6. In that chapter we projected out the vacuum diagrams from the result, while here we will look at the part that is only vacuum diagrams. In particular, we have the following equality, which we will call "Sawon's identity [12]":

$$\langle \Omega', (\frown)^n \rangle = (\frac{1}{24}\ominus)^n. \tag{B.1}$$

Proof. Proceed by induction on n. The result is trivial for $n = 0$.

$$\begin{aligned}
\langle \Omega', (\frown)^n \rangle &= \langle \Omega', \frown \sqcup (\frown)^{n-1} \rangle \\
&= \langle \partial_{\smile}(\Omega'), (\frown)^{n-1} \rangle && \text{by Lemma 5.3} \\
&= \frac{1}{24}\ominus \langle \Omega', (\frown)^{n-1} \rangle && \text{by Lemma 6.2 and explicit computation} \\
&= \left(\frac{1}{24}\ominus\right)^n && \text{by induction}
\end{aligned}$$

$\square$

Equation B.1 is already enough to fix the coefficients in Ω'.

Lemma B.1. *In the Lie algebra $\mathfrak{sl}_2$, we have the following relations:*

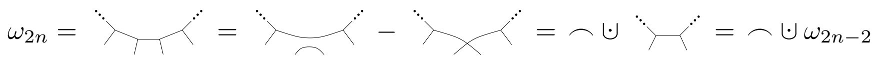
$$\bigcirc \equiv 3 \qquad \asymp \equiv)(- \times$$

Proof. The first relation says that $\mathfrak{sl}_2$ is 3-dimensional. For the second relation, note that both sides, considered as elements of $\mathrm{End}(\mathfrak{sl}_2 \otimes \mathfrak{sl}_2)$, are multiples of the projection onto the antisymmetric part $\mathfrak{sl}_2 \wedge \mathfrak{sl}_2$ (which is 3-dimensional). A little computation fixes the constant. (Note that the constant depends on the metric. Here we use $\langle x, y \rangle = -\operatorname{tr}(xy)$, where the trace is taken in the adjoint representation.) $\square$

Apply the $\mathfrak{sl}_2$ relations above to both sides of Sawon's identity. For the RHS, we find that that $\ominus \equiv 6$. For the LHS, we will use the following two lemmas.

Lemma B.2. *Modulo the $\mathfrak{sl}_2$ relations, $\omega_{2n} \equiv 2(\frown)^n$.*

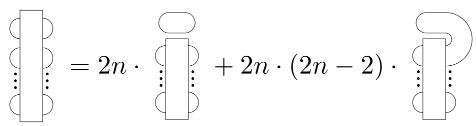
Proof. Proceed by induction. This is a straightforward computation for $n = 1$. For $n > 1$, compute as follows:



□

Lemma B.3. *Modulo the $\mathfrak{sl}_2$ relations, $\langle(\frown)^n, (\frown)^n\rangle = (2n+1)!$.*

Proof. Proceed by induction. The statement is trivial for $n = 0$. For $n > 0$, the two ends of the first strut on the left hand side can either connect to the two ends of a single right hand strut or they can connect to two different struts. These happen in $2n$ and $2n \cdot (2n-2)$ ways, respectively. (Note that there are $2n \cdot (2n-1)$ ways in all of gluing these two legs.) We therefore have



and

$$\begin{aligned}\big\langle(\frown)^n, (\frown)^n\big\rangle &= \big(2n \bigcirc + 2n\cdot(2n-2)\big)\big\langle(\frown^{n-1}, (\frown)^{n-1}\big\rangle \\ &\equiv 2n\cdot(2n+1)\big\langle(\frown^{n-1}, (\frown)^{n-1}\big\rangle \\ &\equiv (2n+1)! \qquad\qquad \text{by induction.}\end{aligned}$$

□

Proposition B.4. $\sum a_n x^n = \frac{1}{2}\log\frac{\sinh(x/2)}{x/2}$.

Proof. By Lemma B.2, we find

$$\Omega' = \exp\Big(\sum_n a_{2n}\omega_{2n}\Big) \equiv \exp\Big(\sum_n 2a_{2n}(\frown)^n\Big).$$

Set $f(x) = \exp(2\sum a_{2n}x^n) = \sum f_n x^n$. Then by Lemma B.3,

$$\begin{aligned}\langle\Omega', (\frown)^n\rangle &\equiv \langle f(\frown), (\frown)^n\rangle = \langle f_n(\frown)^n, (\frown)^n\rangle \equiv f_n(2n+1)! \\ &= \left(\frac{1}{24}\ominus\right)^n \equiv \frac{1}{4^n}.\end{aligned}$$

so

$$\begin{aligned}f_n &= \frac{1}{4^n(2n+1)!} \\ f(x) &= \frac{\sinh(\sqrt{x}/2)}{\sqrt{x}/2} \\ \exp\left(2\sum_n a_n x^n\right) &= \frac{\sinh(x/2)}{x/2} \\ \sum_n a_n x^n &= \frac{1}{2}\log\frac{\sinh(x/2)}{x/2}.\end{aligned}$$

□

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